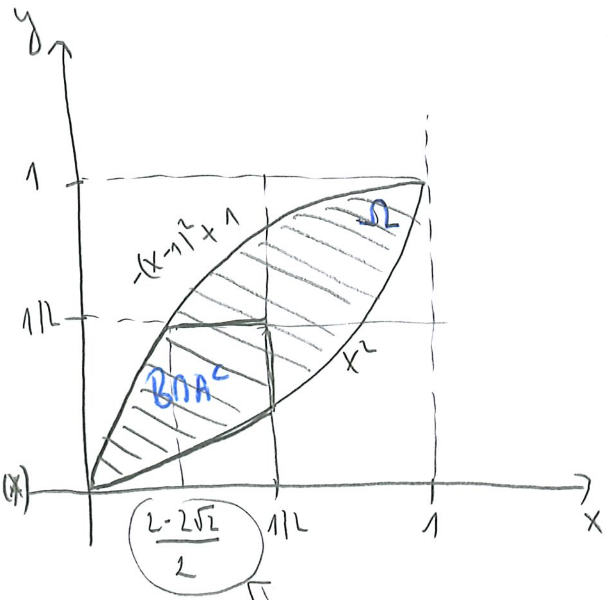
UVIS - ISPITNI ROK 5.9.2022.

1. • $\Omega = \{(x,y)\in\mathbb{R}^2 : x\geq 0,\ y\geq 0,\ x^2 \leq y \leq -(x-1)^2+1\}$

• $A = \{(x,y)\in\Omega : x \geq 1/2\}$

• $B = \{(x,y)\in\Omega : y \leq 1/2\}$



• $P(B|A^c) = ?$

$$P(B|A^c) = \frac{P(B\cap A^c)}{P(A^c)} = \frac{\frac{\lambda(B\cap A^c)}{\lambda(\Omega)}}{\frac{\lambda(A^c)}{\lambda(\Omega)}} = \frac{\lambda(B\cap A^c)}{\lambda(A^c)} \quad (*)$$

$y = 1/2$ & $y = -(x-1)^2+1$

$\Rightarrow x_1 = \frac{2-2\sqrt{2}}{2},\ x_2 = \cancel{\frac{2+2\sqrt{2}}{2}}$

• $$\lambda(A^c) = \int_0^{1/2} \left(-(x-1)^2+1 - x^2\right)dx = \int_0^{1/2}(-2x^2+2x)\,dx = \frac{1}{6}$$

• $$\lambda(B\cap A^c) = \int_0^{(2-2\sqrt{2})/2} \left(-(x-1)^2+1-x^2\right)dx + \int_{\frac{2-2\sqrt{2}}{2}}^{1/2} \left(\frac{1}{2}-x^2\right)dx = \ldots = \frac{9-4\sqrt{2}}{24}$$

$$\overset{(*)}{\Rightarrow} P(B|A^c) = \frac{\frac{9-4\sqrt{2}}{24}}{\frac{1}{6}} = \frac{9-4\sqrt{2}}{4}$$

2. $(\Omega, \mathcal{F}, P)$, $A, B \in \mathcal{F}$ nezavisni

Treba dokazati $P(A\cup B) + (1-\max\{P(A),P(B)\})^2 \leq 1$

$$\Leftrightarrow (1-\max\{P(A),P(B)\})^2 \leq P(A^c\cap B^c)$$

• Krenimo s desnom stranom:

$$P(A^c\cap B^c) \overset{\text{nez.}}{=} P(A^c)\cdot P(B^c) \overset{\text{sup. dog.}}{=} (1-P(A))(1-P(B)) \quad (1)$$

• S druge strane:

$$\left.\begin{array}{l} P(A) \leq \max\{P(A),P(B)\} \\ P(B) \leq \max\{P(A),P(B)\} \end{array}\right. \Rightarrow \left.\begin{array}{l} 1-P(A) \geq 1-\max\{P(A),P(B)\} \\ 1-P(B) \geq 1-\max\{P(A),P(B)\} \end{array}\right. \quad (2)$$

Sada iz (1) & (2) slijedi $P(A^c\cap B^c) = (1-P(A))(1-P(B)) \geq (1-\max\{P(A),P(B)\})^2$ ■

3. $X \sim \mathcal{E}(1) \Rightarrow f_X(x) = \begin{cases} 0, & x<0 \\ e^{-x}, & x \geq 0 \end{cases}, \quad F_X(x) = \begin{cases} 0, & x<0 \\ 1-e^{-x}, & x \geq 0 \end{cases}$

- $Y = 1 - \ln\left(\frac{e^{-X}}{1-e^{-X}}\right)$

1) Funkcija distribucije: ((*) vrijedi jer je $1-e^{-X} > 0$ jer je $P(X>0)=1$)

$$F_Y(y) = P(Y \leq y) = P\left(1 - \ln\left(\frac{e^{-X}}{1-e^{-X}}\right) \leq y\right) = P\left(\ln\left(\frac{e^{-X}}{1-e^{-X}}\right) \geq 1-y\right) =$$

$$= P\left(\frac{e^{-X}}{1-e^{-X}} \geq e^{1-y}\right) \overset{(*)}{=} P\left(e^{-X} \geq e^{1-y} \cdot (1-e^{-X})\right)$$

$$= P\left(e^{-X} \geq \frac{e^{1-y}}{1+e^{1-y}}\right) = P\left(-X \geq \ln\left(\frac{e^{1-y}}{1+e^{1-y}}\right)\right) = P\left(X \leq -\ln\left(\frac{e^{1-y}}{1+e^{1-y}}\right)\right)$$

$$= F_X\left(-\ln\left(\frac{e^{1-y}}{1+e^{1-y}}\right)\right) \overset{(**)}{=} 1 - e^{-\left(-\ln\left(\frac{e^{1-y}}{1+e^{1-y}}\right)\right)} = 1 - \frac{e^{1-y}}{1+e^{1-y}} = \frac{1}{1+e^{1-y}}$$

((**) vrijedi jer je $0 < \frac{e^{1-y}}{1+e^{1-y}} < 1 \Rightarrow \ln\left(\frac{e^{1-y}}{1+e^{1-y}}\right) < 0 \Rightarrow -\ln\left(\frac{e^{1-y}}{1+e^{1-y}}\right) > 0$) $y \in \mathbb{R}$

$$\Rightarrow F_Y(y) = \frac{1}{1+e^{1-y}}, \quad y \in \mathbb{R}$$

2) funkcija gustoće:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(\frac{1}{1+e^{1-y}}\right) = \frac{e^{1-y}}{(1+e^{1-y})^2}, \quad y \in \mathbb{R}$$

4. · X - s.v. kojom modeliramo broj točnih odgovora Ane

Ima ukupno $n=20$ pitanja, a na svakom pitanju postoji $\binom{4}{2}=6$ mogućih odabira (npr. zaokružiti a) i c))

$\Rightarrow X \sim B(20, \frac{1}{6}) \Rightarrow P(X=k)=\binom{20}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{20-k}, \quad k\in\{0,1,\dots,20\}$

· Y - s.v. kojom modeliramo iznos kuna potrošenih na odjeću

$Y=X^2 \Rightarrow R(Y)=\{0,1,4,9,\dots,400\}, \quad P(Y=k^2)=P(X=k)=\binom{20}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{20-k},$

$k\in R(X).$

$$\Rightarrow Y=\begin{pmatrix} 0 & 1 & 4 & \dots & 400 \\ \left(\frac{5}{6}\right)^{20} & 20\cdot\frac{1}{6}\left(\frac{5}{6}\right)^{19} & 190\cdot\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{18} & \dots & \left(\frac{1}{6}\right)^{20}\end{pmatrix}$$

· $E[Y]=?$ $\Rightarrow E[Y]=E[X^2]=\operatorname{Var}X+(E[X])^2=20\cdot\frac{1}{6}\cdot\frac{5}{6}+20\cdot\frac{1}{6}$

$=\frac{55}{9}$ //

5. · X, Y s.v. t.d. vrijedi

$E[X]=0, \quad E[Y]=1, \quad E[XY]=-4, \quad \operatorname{Var}X=2, \quad \operatorname{Var}Y=8$

· Treba dokazati da je $Y=-2X+1$

· Kako je $\rho_{XY}=\frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}X\cdot\operatorname{Var}Y}}=\frac{E[XY]-EX\cdot EY}{\sqrt{\operatorname{Var}X\cdot\operatorname{Var}Y}}=\frac{-4-0\cdot1}{\sqrt{2\cdot8}}=-\frac{4}{4}=\boxed{-1}$

$\Rightarrow X$ i Y su povezane linearnom padajućom vezom, tj. $\underbrace{\boxed{Y=aX+b}}_{(*)}$, $a<0$

$a, b = ?$

· Iz $(*)$ slijedi $EY=aEX+b$, tj. $1=a\cdot0+b \Rightarrow \boxed{b=1}$

i $\operatorname{Var}Y=\operatorname{Var}(aX+b)=a^2\operatorname{Var}X$, tj. $8=a^2\cdot2 \Rightarrow a^2=4 \,/\sqrt{\ }$

$a=-2 \ (a<0)$

$\Rightarrow a=-2$ & $b=1 \Rightarrow \boxed{Y=-2X+1}$ ∎