UVIS, 2. kolokvij, 2017./2018. GRUPA B

2. 3 crvene, 7 zelenih kuglica

izvlačimo tri

a) X_1 - broj izvučenih zelenih kuglica, $X_1 \sim \mathcal{H}(\overset{m}{3}, \overset{M}{7}, \overset{N}{10})$

$$P(X_1 = 2) = \frac{\binom{7}{2}\binom{3}{1}}{\binom{10}{3}} = \frac{21}{40}, \quad E[X] = \frac{mM}{N} = \frac{3 \cdot 7}{10} = \frac{21}{10}$$

b) X_2 - broj izvučenih zelenih kuglica, $X_2 \sim \mathcal{B}(3, \frac{7}{10})$

$$P(X_2 = 2) = \binom{3}{2}\left(\frac{7}{10}\right)^2\left(\frac{3}{10}\right)^1 = 3 \cdot \frac{49}{100} \cdot \frac{3}{10} = \frac{441}{1000}, \quad E[X] = np = 3 \cdot \frac{7}{10} = \frac{21}{10} //$$

3. $P(X = n) = \frac{2}{3^n}, n \in \mathbb{N}, \quad P(|X - E(X)| \geq 2\sigma_X) = ?$

Neka je $g(t) = 2 \cdot \sum_{n=1}^{\infty}\left(\frac{e^t}{3}\right)^n = 2 \cdot \frac{\frac{e^t}{3}}{1 - \frac{e^t}{3}} = \frac{2e^t}{3 - e^t} //$

$$\Rightarrow g'(t) = \frac{6e^t}{(e^t - 3)^2} \quad \& \quad g'(t) = 2\sum_{n=1}^{\infty} n\left(\frac{e^t}{3}\right)^n \quad (*)$$

$$\Rightarrow g''(t) = -\frac{(3 + e^t)6e^t}{(e^t - 3)^3} \quad \& \quad g''(t) = 2\sum_{n=1}^{\infty} n^2\left(\frac{e^t}{3}\right)^n \quad (**)$$

Iz $(*)$ $\Rightarrow$ $g'(0) = \frac{6}{(1-3)^2} = 2\sum_{n=1}^{\infty}\frac{n}{3^n} \iff 2\sum_{n=1}^{\infty}\frac{n}{3^n} = E[X] = \frac{3}{2} //$

Iz $(**)$ $\Rightarrow$ $g''(0) = -\frac{6(3+1)}{(1-3)^3} = 2\sum_{n=1}^{\infty} n^2\left(\frac{e^0}{3}\right)^n \iff 2\sum_{n=1}^{\infty}\frac{n^2}{3^n} = E[X^2] = 3 //$

$$\Rightarrow \operatorname{Var} X = E[X^2] - (E[X])^2 = 3 - \frac{9}{4} = \frac{3}{4} // \quad \Rightarrow \sigma_X = \frac{\sqrt{3}}{2} //$$

$$\Rightarrow P(|X - E[X]| \geq 2\sigma_X) = P\left(\left|X - \frac{3}{2}\right| \geq \sqrt{3}\right) = P\left(X \in \langle -\infty, \underbrace{\tfrac{3}{2} - \sqrt{3}}_{<0}] \cup [\tfrac{3}{2} + \sqrt{3}, +\infty\rangle\right)$$

$$= P\left(X \geq \frac{3}{2} + \sqrt{3}\right) = \sum_{k=4}^{\infty} P(X = k) = \sum_{k=4}^{\infty}\frac{2}{3^k} = \frac{2}{3^4} \cdot \frac{1}{1 - \frac{1}{3}} = \frac{2}{3^4} \cdot \frac{3}{2} = \frac{1}{3^3} = \frac{1}{27} //$$

(4.) $f_X(x) = \begin{cases} \frac{k}{16} x^3 e^{-x/2}, & x > 0 \\ 0, & x \leq 0 \end{cases}$

$\left| \begin{matrix} \frac{x}{2} = t \\ dx = 2dt \end{matrix} \right|$

a) $k = ?$ $\int_{\mathbb{R}} f_X(x) = 1 \Leftrightarrow \int_0^{+\infty} \frac{k}{16} x^3 e^{-x/2} dx = 1 \Leftrightarrow \frac{k}{16} \int_0^{+\infty} (2t)^3 e^{-t} 2 dt = 1$

$\Leftrightarrow k \cdot \int_0^{+\infty} t^3 e^{-t} dt = 1 \Leftrightarrow k \cdot \Gamma(4) = 1 \Rightarrow k = \frac{1}{3!} = \frac{1}{6}$ //

b) $E[Y] = 2E[X] + 1$

$E[X] = \int_0^{+\infty} \frac{1}{96} x^4 e^{-x/2} dx = \left| \begin{matrix} \frac{x}{2} = t \\ dx = 2dt \end{matrix} \right| = \int_0^{+\infty} \frac{1}{96} 2^4 t^4 e^{-t} 2 dt = \frac{1}{3} \int_0^{+\infty} t^4 e^{-t} dt$

$= \frac{1}{3} \Gamma(5) = \frac{4!}{3} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{3} = 8$ //

$\Rightarrow E[Y] = 2 \cdot 8 + 1 = 17$ //

(5.) $X \sim U(1, 3)$
$Y = e^X$
$\Rightarrow f_X(x) = \begin{cases} \frac{1}{2}, & x \in (1, 3) \\ 0, & \text{inače} \end{cases}$, $F_X(x) = \begin{cases} 0, & x \in (-\infty, 1) \\ \frac{x-1}{2}, & x \in [1, 3) \\ 1, & x \in [3, +\infty) \end{cases}$

$y > 0: F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln y) = F_X(\ln y)$

$= \begin{cases} 0, & \ln y \in (-\infty, 1) \\ \frac{\ln y - 1}{2}, & \ln y \in [1, 3) \\ 1, & \ln y \in [3, +\infty) \end{cases} = \begin{cases} 0, & y \in (-\infty, e) \\ \frac{\ln y - 1}{2}, & y \in [e, e^3) \\ 1, & y \in [e^3, +\infty) \end{cases}$ //

$\Rightarrow f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{1}{2y}, & y \in [e, e^3) \\ 0, & \text{inače} \end{cases}$ //

6. 7 žutih po 5g & 3 zelene kuglice po 4g.

→ izvlačimo odjednom 2 kuglice bez vraćanja.

X - ukupna težina izvučenih kuglica, Y - broj izvučenih zelenih kuglica.

$\mathcal{R}(X) = \{8g, 9g, 10g\}$, $\mathcal{R}(Y) = \{0, 1, 2\}$

X\Y	0	1	2	
8	0	0	3/45	3/45
9	0	7/15	0	7/15
10	7/15	0	0	7/15
	7/15	7/15	3/45	

$P(X=8, Y=0) = 0$

$P(X=8, Y=1) = 0$

$P(X=8, Y=2) = \dfrac{\binom{3}{2}}{\binom{10}{2}} = \dfrac{3}{45}$

$P(X=9, Y=0) = 0$

$P(X=9, Y=1) = \dfrac{\binom{3}{1}\binom{7}{1}}{\binom{10}{2}} = \dfrac{7}{15}$

$P(X=9, Y=2) = 0$

$P(X=10, Y=0) = \dfrac{\binom{7}{2}}{\binom{10}{2}} = \dfrac{7}{15}$

$P(X=10, Y=1) = 0$

$P(X=10, Y=2) = 0$

$\Rightarrow X = \begin{pmatrix} 8 & 9 & 10 \\ 3/45 & 7/15 & 7/15 \end{pmatrix}$

$Y = \begin{pmatrix} 0 & 1 & 2 \\ 7/15 & 7/15 & 3/45 \end{pmatrix}$

Kako je $P(X=8, Y=0) = 0$
$P(X=8) \cdot P(Y=0) = \dfrac{3}{45} \cdot \dfrac{7}{15}$ $\Rightarrow P(X=8, Y=0) \neq P(X=8) \cdot P(Y=0)$
$\Rightarrow$ X i Y nisu nezavisne!

$P(X \cdot Y < 10) = ?$

$\mathcal{R}(X \cdot Y) = \{0, 8, 16, 9, 18, 10, 20\} = \{0, 8, 9, 10, 16, 18, 20\}$

$P(X \cdot Y < 10) = P(X \cdot Y = 0) + P(X \cdot Y = 8) + P(X \cdot Y = 9)$

$= P(X=8, Y=0) + P(X=9, Y=0) + P(X=10, Y=0) + P(X=8, Y=1) + P(X=9, Y=1)$

$= 0 + 0 + 7/15 + 0 + 7/15 = 14/15$ //