UVIS - drugi kolokvij, 2020./2021.

2. • X – sl. varijabla kojom modeliramo broj bacanja kockice (do pojave četvorke)

↳ $X \sim G(\frac{1}{6})$, $\mathcal{R}(X) = \mathbb{N}$, $P(X=k) = (1-\frac{1}{6})^{k-1}(\frac{1}{6}) = (\frac{5}{6})^{k-1} \cdot \frac{1}{6}$, $k \in \mathbb{N}$

• Y – sl. varijabla kojom modeliramo dobitak/gubitak u igri

↳ $Y = 10 \cdot X - 50$

a) $E[Y] = ?$ $\Rightarrow E[Y] = 10\,E[X] - 50 = 10 \cdot \frac{1}{1/6} - 50 = 10$ kn //

b) $P(Y=20) = P(10X - 50 = 20) = P(X=7) = (\frac{5}{6})^6 \cdot \frac{1}{6}$

3. $$f(x) = \begin{cases} 5a^5 x^{-6}, & x \geq 3 \\ 0, & x < 3 \end{cases}$$

a) $a = ?$

1° nenegativnost: $f(x) \geq 0 \ \forall x \in \mathbb{R} \Rightarrow \boxed{a \geq 0}$ ✓

2° normiranost: $\int_{\mathbb{R}} f(x)dx = 1 \Rightarrow \int_3^{+\infty} 5a^5 x^{-6} dx = 1 \Rightarrow 5a^5 \cdot \frac{x^{-5}}{-5}\Big|_3^{+\infty} = 1 \Rightarrow a^5 \cdot 3^{-5} = 1 \Rightarrow \boxed{a = 3}$ ✓

b) $E[Y] = 4\,E[X] + 3$; $E[X] = \int_{\mathbb{R}} x f(x) dx = \int_3^{+\infty} 5 \cdot 3^5 \cdot x^{-6} \cdot x\, dx = 5 \cdot 3^5 \cdot \frac{x^{-4}}{-4}\Big|_3^{+\infty} = \frac{15}{4}$ //

$\Rightarrow E[Y] = 4 \cdot \frac{15}{4} + 3 = 15 + 3 = 18$ //

4. $\mathcal{R}(X) = \mathbb{N}_0$ & $g_X(z) = e^{2z-2} \Rightarrow g_X'(z) = 2e^{2z-2}$; $g_X''(z) = 4e^{2z-2}$

a) $E[X] = g_X'(1) = 2e^{2 \cdot 1 - 2} = \boxed{2}$ //

b) $\mathrm{Var}\,X = g_X''(1) + g_X'(1) - g_X'(1)^2 = 4e^{2\cdot 1-2} + 2e^{2 \cdot 1 - 2} - (2e^{2\cdot 1-2})^2 = 4 + 2 - 4 = \boxed{2}$ //

$\Rightarrow \mathrm{Var}\,Y = 4\,\mathrm{Var}\,X = 8$ //

c) $P(X=0) = ?$ Kako je $g_X(z) = \sum_{k=0}^{\infty} p_k z^k \Rightarrow g_X(0) = p_0 = P(X=0)$

$\Rightarrow P(X=0) = e^{2 \cdot 0 - 2} = \boxed{e^{-2}}$ //

5.

$$X \sim \mathcal{E}(2) \Rightarrow f_X(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$Y = 2e^X \Rightarrow g(x) = 2e^x \text{ (bijekcija)} \Rightarrow (g \circ g^{-1})(x) = x \Rightarrow g^{-1}(x) = \ln\left(\frac{x}{2}\right)$$

$$\Rightarrow [g^{-1}(x)]' = \frac{1}{\frac{x}{2}} \cdot \frac{1}{2} = \boxed{\frac{1}{x}}$$

$$\Rightarrow \cdot f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot |[g^{-1}(y)]'|, & y \in \mathcal{R}(g) \\ 0, & y \in \mathcal{R}(g)^c \end{cases}$$

$$= \begin{cases} f_X(\ln(\frac{y}{2})) \cdot \left|\frac{1}{y}\right|, & y \in \mathcal{R}(g) \\ 0, & y \in \mathcal{R}(g)^c \end{cases} = \begin{cases} 2e^{-2 \cdot \ln(\frac{y}{2})} \cdot \left|\frac{1}{y}\right|, & \ln(\frac{y}{2}) \geq 0 \\ 0, & \ln(\frac{y}{2}) < 0 \end{cases}$$

$$= \begin{cases} 2 \cdot \left(\frac{y}{2}\right)^{-2} \cdot \frac{1}{y}, & y \geq 2 \\ 0, & y < 2 \end{cases} = \begin{cases} \left(\frac{2}{y}\right)^3, & y \geq 2 \\ 0, & y < 2 \end{cases} //$$

$$\cdot E[Y] = \int_{\mathbb{R}} y f_Y(y) dy = \int_2^{+\infty} y \cdot \left(\frac{2}{y}\right)^3 dy = 8 \int_2^{+\infty} y^{-2} dy = 8 \left.\frac{y^{-1}}{-1}\right|_2^{+\infty} = \boxed{4} //$$

6.

X – broj parnih brojeva koji padnu pri bacanju kockice $\Rightarrow \mathcal{R}(X) = \{0, 1, 2\}$

Y – broj glava koje padnu pri bacanju novčića $\Rightarrow \mathcal{R}(Y) = \{0, 1, 2\}$

• tablica distribucije:

X i Y nezavisne & $X \sim \mathcal{B}(2, \frac{1}{2}), Y \sim \mathcal{B}(2, \frac{1}{2})$

$$\Rightarrow P(X = i, Y = j) = \binom{2}{i}\left(\frac{1}{2}\right)^i\left(\frac{1}{2}\right)^{2-i} \cdot \binom{2}{j}\left(\frac{1}{2}\right)^j \cdot \left(\frac{1}{2}\right)^{2-j}$$

$$= \binom{2}{i}\binom{2}{j}\left(\frac{1}{2}\right)^4$$

X\Y	0	1	2	M_X
0	1/16	1/8	1/16	1/4
1	1/8	1/4	1/8	1/2
2	1/16	1/8	1/16	1/4
M_Y	1/4	1/2	1/4	1

• marginalne dist:

$$X \stackrel{d}{=} Y = \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

• $\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var X \cdot Var Y}} \stackrel{nez}{=} 0 \Rightarrow$ X i Y su nekorelirane!

• $P(X \cdot Y = 4) = P(X = 2, Y = 2) = 1/16$ //