A fast algorithm for solving the multiple ellipse detection problem

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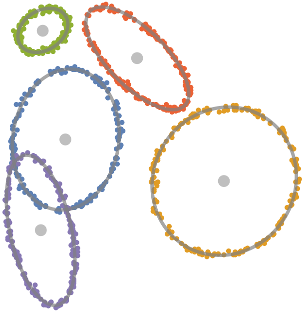
Methodological framework for efficient energy management by intelligent data analytics IP-2016-06-8350

Problem statement

$\mathcal{A} = \{a^i = (x_i, y_i)^T \in \mathbb{R}^2 \colon \alpha_1 \le x_i \le \beta_1,\ \alpha_2 \le y_i \le \beta_2, i = 1, \ldots, m\}$

$\mathcal{A} \subset [\alpha, \beta] := [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \subset \mathbb{R}^2,\ \alpha = (\alpha_1, \alpha_2),\ \beta = (\beta_1, \beta_2)$

(set of data points coming from a number of ellipses in the plane not known in advance that should be reconstructed or detected)



Applications

Pattern recognition and computer vision:

C.Akinlar, C.Topal, *Pattern Recognition* **46**(2013)

D.K.Prasad, M.K.H.Leung, C.Quek, *Pattern Recognition* **46**(2013)

Applications in medicine, robotics, object detection;

R.Grbić, D.Grahovac, R.Scitovski, *Pattern Recognition* **60**(2016)

H.Dong, D.K.Prasad, I-M.Chen, *Pattern Recognition* **81**(2018)

P.Meissner et al., *Robotics and Autonomous Systems* **62**(2014)

Wireless sensor networks;

M.Moshtaghi et al., *Pattern Recognition* **44**(2011)

Agriculture:

M.A.Kashiha, C.Bahr, S.Ott, C.P.Moons, T.A.Niewold, F.T.D.Berckmans, *Livestock Science* **159**(2014)

Astronomical and geological shape segmentation:

T.J.Wynna S.A.Stewart, *Journal of Structural Geology*, **27**(2005)

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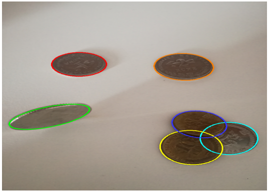
Real world images detection

(a) Original

(c) Detection



(a) Original

(c) Detection

Methods and algorithms

- Methods based on Hough transform:
  H.Dong, D.K.Prasad, I-M.Chen, *Pattern Recognition* **81**(2018)
- EDCircles algorithm:
  Akinlar, C.Topal, *Pattern Recognition* **46**(2013)
- Application of Least Squares and RANSAC method:
  R.Grbić, D.Grahovac, R.Scitovski, *Pattern Recognition* **60**(2016)
- Least squares based geometric ellipse fitting method:
  D.K.Prasad, M.K.H.Leung, C.Quek, *Pattern Recognition* **46**(2013)
- Three-point algorithm with edge angle information:
  B-K.Kwon et al., *International Journal of Control, Automation and Systems* **14**(2016)
- . . .

Problem statement - center-based clustering

$\mathcal{A} = \{a_i \in \mathbb{R}^n : i = 1, \ldots, m\} \subset [\alpha, \beta] \subset \mathbb{R}^n$, (data set)

$\Pi(\mathcal{A}) = \{\pi_1, \ldots, \pi_k\}$ (k-partition)

$$\bigcup_{i=1}^{k} \pi_i = \mathcal{A}, \qquad \pi_r \cap \pi_s = \emptyset, \quad r \neq s, \qquad |\pi_j| \geq 1, \quad j = 1, \ldots, k$$

$\mathcal{P}(\mathcal{A}; k)$ (set of all partitions)

$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, $d(u, v) = \|u - v\|_2^2$ (LS distance-like function)

$c_j := \operatorname*{argmin}_{x \in \operatorname{conv}(\mathcal{A})} \sum_{a^i \in \pi_j} d(x, a^i)$ (centroid)

Global optimization problem (GOP)

$$\operatorname*{argmin}_{\Pi \in \mathcal{P}(\mathcal{A}; k)} \mathcal{F}(\Pi), \qquad \mathcal{F}(\Pi) = \sum_{j=1}^{k} \sum_{a^i \in \pi_j} d(c_j, a^i)$$

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Problem statement - center-based clustering

Given are centers $c_1, \dots, c_k \in [\alpha, \beta]$

Minimal distance principle gives the partition $\Pi = \{\pi(c_1), \dots, \pi(c_k)\}$

$$\pi(c_j) = \{a \in \mathcal{A} : d(c_j, a) \leq d(c_s, a), \ \forall s = 1, \dots, k\}, \qquad j = 1, \dots, k.$$

Global optimization problem:

$$\operatorname*{argmin}_{c \in [\alpha,\beta]^k} F(c), \qquad F(c) = \sum_{i=1}^{m} \min_{1 \leq j \leq k} d(c_j, a^i).$$

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Problem statement - an ellipse

$$P(x,y)=0, \quad P(x,y)=Ax^2+2Bxy+Cy^2+Dx+Ey+F, \quad AC-B^2>0$$

$$ax^2+2bxy+cy^2+dx+ey+d=0, \quad ac-b^2=1.$$

$$\underset{\substack{a,b,c,d,e,f\in\mathbb{R}\\ ac-b^2=1}}{\operatorname{argmin}} \sum_{i=1}^{m}(ax_i^2+2bx_iy_i+cy_i^2+dx_i+ey_i+f)^2 \quad \text{(Linear least squares problem)}$$

Ellipse:

$$E(S,\xi,\eta,\vartheta)=\{(x,y)\in\mathbb{R}^2\colon \frac{[(x-p)\cos\vartheta+(y-q)\sin\vartheta]^2}{\xi^2}+\frac{[(x-p)\sin\vartheta-(y-q)\cos\vartheta]^2}{\eta^2}=1\}$$

$S=(p,q)^T$: center of the ellipse;
$\xi,\eta>0$: lengths of semi-axes;
ϑ: angle between the semi-axes ξ and a positive direction of the abscissa.

Parametric form:

$$\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}=S+U(\vartheta)\begin{bmatrix}\xi\cos t\\ \eta\sin t\end{bmatrix}, \quad U(\vartheta)=\begin{bmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{bmatrix}, \quad t\in[0,2\pi]$$

Problem statement - an ellipse

$$P(x,y)=0,\quad P(x,y)=Ax^2+2Bxy+Cy^2+Dx+Ey+F,\quad AC-B^2>0$$

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Problem statement - an ellipse

$$P(x,y)=0, \quad P(x,y)=Ax^2+2Bxy+Cy^2+Dx+Ey+F, \quad AC-B^2>0$$

$$ax^2+2bxy+cy^2+dx+ey+d=0, \quad ac-b^2=1.$$

$$\underset{\substack{a,b,c,d,e,f\in\mathbb{R}\\ ac-b^2=1}}{\operatorname{argmin}} \sum_{i=1}^{m}(ax_i^2+2bx_iy_i+cy_i^2+dx_i+ey_i+f)^2$$ (Linear least squares problem)

Ellipse:

$$E(S,\xi,\eta,\vartheta)=\{(x,y)\in\mathbb{R}^2 : \frac{[(x-p)\cos\vartheta+(y-q)\sin\vartheta]^2}{\xi^2}+\frac{[(x-p)\sin\vartheta-(y-q)\cos\vartheta]^2}{\eta^2}=1\}$$

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Problem statement - an ellipse as a Mahalanobis circle

Ellipse:

$$E(S,\xi,\eta,\vartheta)=\{u\in\mathbb{R}^2\colon (S-u)^T\Sigma^{-1}(S-u)=1\},\qquad \Sigma=U(\vartheta)\begin{bmatrix}\xi^2 & 0\\ 0 & \eta^2\end{bmatrix}U^T(\vartheta)$$

It can be written:

$$\sqrt{\det\Sigma}(S-u)^T\Sigma^{-1}(S-u)=r^2,\qquad r^2:=\xi\eta=\sqrt{\det\Sigma}$$

Definition 1

Mahalanobis circle (M-circle) $\mathfrak{M}(S,r;\Sigma)$ with the center at the point $S\in\mathbb{R}^2$, radius r and the covariance matrix Σ, is the set of points

$$\mathfrak{M}(S,r;\Sigma)=\{u\in\mathbb{R}^2\colon d_M(S,u;\Sigma)=r^2\},$$

where $d_M\colon \mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}_+$ given by

$$d_M(S,u;\Sigma)=\sqrt{\det\Sigma}(S-u)^T\Sigma^{-1}(S-u)=\|u-v\|_\Sigma^2$$

is the *Mahalanobis distance-like function*.

$\mathfrak{M}(S,1;\Sigma)=\{u\in\mathbb{R}^2\colon d_M(S,u;\Sigma)=1\}$ normalized Mahalanobis circle ($\xi\cdot\eta=1$)

Problem statement - an ellipse as a Mahalanobis circle

Ellipse:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = S + U(\vartheta) \begin{bmatrix} \xi \cos t \\ \eta \sin t \end{bmatrix}, \quad U(\vartheta) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}, \quad t \in [0, 2\pi] \qquad (\star)$$

M-circle:

$$\mathfrak{M}(S, r; \Sigma) = \{u \in \mathbb{R}^2 \colon d_M(S, u; \Sigma) = r^2\}, \qquad r^2 := \xi\eta = \sqrt{\det \Sigma}$$

Lemma 1

Ellipse $E(S, \xi, \eta, \vartheta)$ given by $(\star)$, can be considered as an M-circle $\mathfrak{M}(S, r; \Sigma)$, with $r = \sqrt{\xi\eta}$. Conversely, an M-circle $\mathfrak{M}(S, r; \Sigma)$ is an ellipse $E(S, \xi, \eta, \vartheta)$, where $\mathrm{diag}(\xi^2, \eta^2) = U^T(\vartheta)\left(\frac{r^2}{\sqrt{\det \Sigma}}\Sigma\right)U(\vartheta)$, $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$, $U(\vartheta) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$ and

$$\vartheta = \tfrac{1}{2} \arctan \tfrac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}} \in \begin{cases} [0, \pi/4), & \sigma_{12} \geq 0 \;\&\; \sigma_{11} > \sigma_{22} \\ [\pi/4, \pi/2), & \sigma_{12} > 0 \;\&\; \sigma_{11} \leq \sigma_{22} \\ [\pi/2, 3\pi/4), & \sigma_{12} \leq 0 \;\&\; \sigma_{11} < \sigma_{22} \\ [3\pi/4, \pi), & \sigma_{12} < 0 \;\&\; \sigma_{11} \geq \sigma_{22} \end{cases}$$

Problem statement - an ellipse as a Mahalanobis circle

Example 1

Let $O = (0,0)^T$ *and* $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1/2 \end{bmatrix}$. *A least squares unit circle* $\{u \in \mathbb{R}^2 \colon \|u\|_2^2 = 1\}$ *with the center in* O *is shown in Fig. 1a, and an M-circle* $\mathfrak{M}(O, \sqrt[4]{\frac{3}{2}}; \Sigma)$ *is shown in Fig. 1b.*

Let $T_1 = (\sqrt[4]{6}, 0)^T$ *and* $T_2 = (0, \frac{1}{\sqrt[4]{6}})^T$ *be the points in the plane. Note that the LS distances from points* T_1 *and* T_2 *to the origin* O *are* $d_{LS}(O, T_1) = \sqrt{6}$ *and* $d_{LS}(O, T_2) = \frac{1}{\sqrt{6}}$, *respectively. At the same time, these points lie on the corresponding M-circle* $\mathfrak{M}(O, \sqrt[4]{\frac{3}{2}}, \Sigma)$ *because* $d_M(O, T_1, \Sigma) = d_M(O, T_2, \Sigma) = 1$ *(see Fig. 1).*

(a) $d_{LS}(O, T_1) = \sqrt{6}$, $d_{LS}(O, T_2) = \frac{1}{\sqrt{6}}$ (b) $d_M(O, T_1, \Sigma) = d_M(O, T_2, \Sigma) = 1$

Fig.1: Least squares and Mahalanobis distance

Problem statement

Multiple ellipse detection problem:

$$\operatorname*{argmin}_{\substack{S\in[\alpha,\beta]^k\\ \xi,\eta\in[0,R]^k,\vartheta\in[0,\pi]^k}} F(S,\xi,\eta,\vartheta), \quad F(S,\xi,\eta,\vartheta)=\sum_{i=1}^{m}\min_{1\le j\le k}\mathfrak{D}(E_j(S_j,\xi_j,\eta_j,\vartheta_j),a^i)$$

Distances from a point a to the ellipse, i.e. an M-circle:

$$\mathfrak{D}_1(\mathfrak{M}(S,r,\Sigma),a)=|\|S-a\|_\Sigma-r| \quad \text{(LAD-distance)},$$
$$\mathfrak{D}_2(\mathfrak{M}(S,r,\Sigma),a)=(\|S-a\|_\Sigma-r)^2 \quad \text{(TLS-distance)},$$
$$\mathfrak{D}(\mathfrak{M}(S,r,\Sigma),a)=(|\|S-a\|_\Sigma^2-r^2)^2 \quad \text{(algebraic distance)}.$$

Multiple ellipse detection problem:

$$\operatorname*{argmin}_{\substack{S\in[\alpha,\beta]^k\\ r\in[0,R]^k,\Sigma\in M_2^k}} F(S,r,\Sigma), \quad F(S,r,\Sigma)=\sum_{i=1}^{m}\min_{1\le j\le k}\mathfrak{D}(\mathfrak{M}_j(S_j,r_j,\Sigma_j),a^i)$$

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Multiple ellipse detection problem:

$$\underset{\substack{S\in[\alpha,\beta]^k\\ \xi,\eta\in[0,R]^k,\vartheta\in[0,\pi]^k}}{\operatorname{argmin}} F(S,\xi,\eta,\vartheta), \quad F(S,\xi,\eta,\vartheta)=\sum_{i=1}^{m}\min_{1\le j\le k}\mathfrak{D}(E_j(S_j,\xi_j,\eta_j,\vartheta_j),a^i)$$

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Multiple ellipse detection problem:

$$\underset{\substack{S\in[\alpha,\beta]^k\\ r\in[0,R]^k,\Sigma\in M_2^k}}{\operatorname{argmin}} F(S,r,\Sigma), \quad F(S,r,\Sigma)=\sum_{i=1}^{m}\min_{1\le j\le k}\mathfrak{D}(\mathfrak{M}_j(S_j,r_j,\Sigma_j),a^i)$$

Possibilities:

(i) Application of globally optimization algorithm `DIRECT` [1-2];

(ii) Modification of *k*-means algorithm [3-4];

(iii) Incremental algorithm [5-6];

- Our proposal: a combination of (i) and (ii).

[1] D.R.Jones, C.D.Perttunen, B.E.Stuckman, *Journal of Optimization Theory and Applications* **79**(1993)
[2] R.Grbić, E.K.Nyarko, R.Scitovski, *Journal of Global Optimization* **57**(2013)
[3] T.Marošević, R.Scitovski, *Croatian Operational Research Review* **6**(2015)
[4] R.Grbić, D.Grahovac, R.Scitovski, *Pattern Recognition* **60**(2016)
[5] R.Scitovski, T.Marošević, *Pattern Recognition Letters* **52**(2014)
[6] A.M.Bagirov, J.Ugon, D.Webb, *Pattern Recognition* **44**(2011)

Construction of good initial approximation

Initial approximation is obtained by solving simple GOP:

$$S^0 \in \operatorname*{argmin}_{S\in[\alpha,\beta]^k} \Phi(S), \quad \Phi(S) = \sum_{i=1}^{m} \min_{1\le j\le k} \mathfrak{D}(\mathcal{C}_j(S_j,1), a^i),$$

where:

$\mathcal{C}_j(S_j,1)$ (unit circle)

$\mathfrak{D}(\mathcal{C}_j(S_j,1), a^i) = (\|S_j - a^i\|^2 - 1)^2$ (algebraic distance)

Initial approximation for GOP $\operatorname*{argmin}_{\substack{S\in[\alpha,\beta]^k \\ r\in[0,R]^k, \Sigma\in M_2^k}} \sum_{i=1}^{m} \min_{1\le j\le k} \mathfrak{D}(\mathfrak{M}_j(S_j, r_j, \Sigma_j), a^i)$ is

$(S_j^0, 1, \mathbf{I})$, $j = 1, \ldots, k$

Adaptive Mahalanobis k-closest M-circle-centers algorithm

Algorithm 1

Step A: (Assignment step) For each set of mutually different M-circles $\mathfrak{M}_1(S_1, r_1, \Sigma_1)$, $\ldots$, $\mathfrak{M}_k(S_k, r_k, \Sigma_k)$, the set $\mathcal{A}$ should be divided into k disjoint unempty clusters $\pi_1, \ldots, \pi_k$ by using the minimal distance principle:

$$\pi_j := \pi_j(E_j) = \{a \in \mathcal{A} \colon \mathfrak{D}(E_j(S_j, r_j, \Sigma_j), a) \leq \mathfrak{D}(E_s(S_s, r_s, \Sigma_s), a),\ \forall s \neq j\};$$

Step B: (Update step) Given a partition $\Pi\{\pi_1, \ldots, \pi_k\}$ of the set $\mathcal{A}$, one can define the corresponding M-circle-centers $\hat{\mathfrak{M}}_j(\hat{S}_j, \hat{r}_j, \hat{\Sigma}_j)$ $j = 1, \ldots, k$ by solving the following optimization problems

$$\underset{\{S_j \leftarrow S_j^0\},\, \{r_j \leftarrow r_j^0\},\, \{\Sigma_j \leftarrow \Sigma_j^0\}}{\operatorname{argmin}} \sum_{a \in \pi_j} \mathfrak{D}(\mathfrak{M}(S_j, r_j, \Sigma_j), a).$$

Initial approximation in Step B:

$$S_j^0 = \frac{1}{|\pi_j|} \sum_{a \in \pi_j} a; \qquad \Sigma_j^0 = \frac{1}{|\pi_j|} \sum_{a \in \pi_j} (S_j^0 - a)(S_j^0 - a)^T; \qquad (r_j^0)^2 = \frac{1}{|\pi_j|} \sum_{a \in \pi_j} \|S_j^0 - a\|^2;$$

$$\sum_{a \in \pi_j} \left(\|S_j^0 - a\|^2 - r^2\right)^2 \geq \sum_{a \in \pi_j} \left(\|S_j^0 - a\|^2 - (r_j^0)^2\right)^2, \quad \text{for all } r \in \mathbb{R}.$$

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New index

Assumptions::

- If $\Pi^\star = \{\pi_1^\star, \dots, \pi_k^\star\}$ is a globally optimal partition with M-circle-centers $E_1^\star, \dots, E_k^\star$, then for a cluster $\pi_j^\star$ the mean squared deviation of its points from M-circle-center $E_j^\star$ $\left(\frac{1}{|\pi_j^\star|} \sum\limits_{a \in \pi_j^\star} \mathfrak{D}(E_j^\star, a) \right)$ is the smallest possible;
- The density around the M-circle-center $E_j^\star$ $\left(\frac{|\pi_j^\star|}{|E_j^\star|} \right)$ is the largest.

$$|E_j^\star| = 4 \int_0^{\pi/2} \sqrt{\xi^2 \cos^2 t + \eta^2 \sin^2 t}\, dt \quad \text{(elliptic integral of the second kind)}$$

Ramanujan approximation:

$$|E_j^\star| \approx \pi(\xi + \eta)\left(1 + \frac{3h}{10 + \sqrt{4 - 3h}}\right), \quad h = \frac{(\xi - \eta)^2}{(\xi + \eta)^2}$$

New Geometrical Objects-index (GO-index):

$$\mathtt{GO}(k) := k \sum_{j=1}^{k} \frac{|E_j^\star|}{|\pi_j^\star|^2} \sum_{a \in \pi_j^\star} \mathfrak{D}(E_j^\star, a)$$

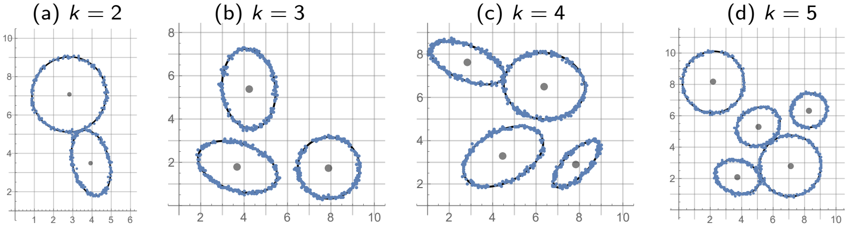
Construction of ellipses $E_j(S_j, \xi_j, \eta_j, \vartheta_j)$, $j = 1, \ldots, k$, $k \in \{2, 3, 4, 5\}$

$S_1, \ldots, S_k \in [1.5, 8.5]^2 \subset [0, 10]^2$ (k random ellipse-centers)

$\xi_j, \eta_j \in \mathcal{U}(0.5, 2)$ (semiaxes)

$\vartheta_j \in \mathcal{U}(0, \pi)$ (angle of rotation)

On the ellipse E_j, we choose $m_j \in \mathcal{U}(180, 220)$ uniformly distributed points and at each point in the direction of the normal vector we take a random point $a^i \in \mathcal{N}(0, \sigma^2)$, $\sigma^2 = .05$



Searching for an optimal k-partition

Algorithm 2

Input: $\mathcal{A} \subset [\alpha, \beta]^2$ {Set of data points}; $k \geq 1$;

1: By using the `DIRECT` algorithm find and initial approximation S^0;

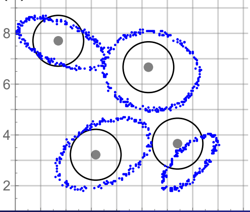
2: Define initial approximation $(S_j^0, 1, \mathbf{I})$, $j = 1, \ldots, k$ and apply the `KMCC` algorithm.
Denote the solution by $\mathfrak{M}_j^\star(S_j^\star, r_j^\star, \Sigma_j^\star)$, $j = 1, \ldots, k$;
Denote the corresponding optimal partition by $\Pi^\star$ and the objective function value by $F^\star$;

3: By using Lemma 1 determine the ellipses $E_j^\star(S_j^\star, \xi_j^\star, \eta_j^\star, \vartheta_j^\star)$, $j = 1, \ldots, k$;

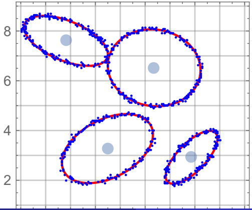
4: Compute value $\texttt{GO}(k)$;

Output: $\{k, \Pi^\star, F^\star, \texttt{GO}(k), E_j^\star(S_j^\star, \xi_j^\star, \eta_j^\star, \vartheta_j^\star), j = 1, \ldots, k\}$.

(a) Initial approximation



(b) Solution



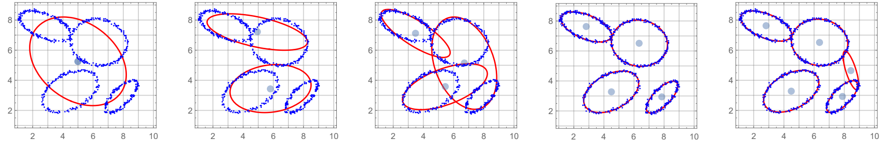
Algorithm 3

Input: $\mathcal{A} \subset [\alpha, \beta]^2$ {Set of data points}; $\epsilon > 0$;

1: Set $r = 1$ and call Algorithm 2;
2: Set $r = r + 1$ and call Algorithm 2;
3: **if** $(F_{r-1} - F_r)/F_1 > \epsilon$, **then**
4: Set $r = r + 1$ and call Algorithm 2 and go to Step 3;
5: **end if**
6: Find a k-partition $\Pi^\star$ $(k \leq r)$ with the smallest GO-index and denote corresponding cluster-centers by $E_1^\star, \dots, E_k^\star$;

Output: $\{k, \Pi^\star, \{E_1^\star, \dots, E_k^\star\}\}$.

(a) $F_1 = 23117.7$, $\mathtt{GO}(1) = 193.7$
(b) $F_2 = 2808.8$, $\mathtt{GO}(2) = 25.7$
(c) $F_3 = 671.6$, $\mathtt{GO}(3) = 2.4$
(d) $F_4 = 19.1$, $\mathtt{GO}(4) = 0.00013$
(e) $F_5 = 18.7$, $\mathtt{GO}(5) = 0.0014$



Searching for an optimal partition with the most appropriate number of clusters

Use of GO-index monotonicity

Algorithm 4

Input: $\mathcal{A} \subset [\alpha, \beta]^2$ {Set of data points};

Set $k = 1$ and call Algorithm 2;
Set $k = k + 1$ and call Algorithm 2;
while $(\texttt{GO}(k) < \texttt{GO}(k-1))$ **do**
    Set $k = k + 1$ and call Algorithm 2;
end while

Output: $\{k, \Pi^\star, \{E_1^\star, \ldots, E_k^\star\}\}$.

Testing the method and the algorithm in the case of separated ellipses

No. of ellipses	Ellipses detected 0	1	2	3	4	5	6	CPU-time (DIRECT)	(k-means)	Total
2	0	0	20	0	0	0	0	0.01166	0.03388	0.04554
3	0	0	0	25	0	0	0	0.03963	0.08338	0.12301
4	1	0	0	0	26	0	0	0.09531	0.11201	0.20732
5	5	1	0	1	0	21	0	0.09542	0.15709	0.25251

Characteristics of Algorithm 4 in the case of separated ellipses

No. of ellipses	Ellipses detected 0	1	2	3	4	5	6	CPU-time (DIRECT)	(k-means)	Total
2	0	0	20	0	0	0	0	0.01544	0.03781	0.05325
3	0	0	0	25	0	0	0	0.03723	0.08193	0.11916
4	0	0	0	0	27	0	0	0.09917	0.11905	0.21822
5	0	0	0	1	0	27	0	0.11779	0.17902	0.29681

Characteristics of Algorithm 3 in the case of separated ellipses

Testing the method and the algorithm in the case of separated ellipses

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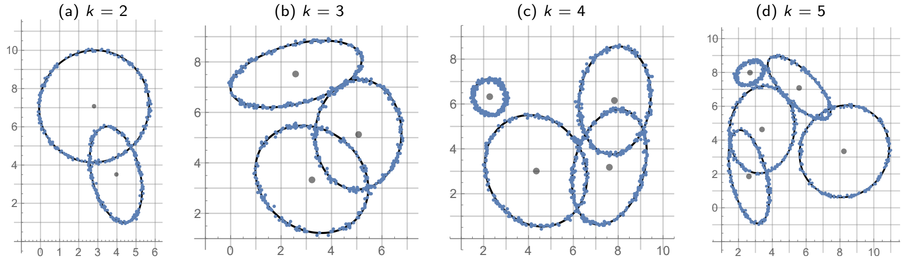
Characteristics of Algorithm 3 in the case of separated ellipses



No. of ellipses	Ellipses detected							CPU-time		
	0	1	2	3	4	5	6	(DIRECT)	(k-means)	Total
2	0	1	19	0	0	0	0	0.01221	0.06475	0.07696
3	0	1	3	21	0	0	0	0.04443	0.15606	0.20049
4	8	2	2	2	13	0	0	0.07108	0.19019	0.26127
5	5	3	3	8	1	8	0	0.09005	0.25748	0.34753

Characteristics of Algorithm 4 in the case of intersecting ellipses

Remark 1

When searching for an optimal k-partition and determining the corresponding GO-index for every cluster $\pi_j^\star$ we can determine the following indicator:

$$\frac{|E_j^\star|}{|\pi_j^\star|^2} \sum_{a\in\pi_j^\star} \mathfrak{D}(E_j^\star, a) \qquad (\star)$$

A small value of number $(\star)$ will indicate if the ellipse-center $E_j^\star$ has been recognized as one of the ellipses searched for. In that case, similarly to [1], cluster $\pi_j^\star$ can be taken out from the set $\mathcal{A}$ and the algorithm should be run again with the rest of the data on a new, simpler set.

As shown in [2], for solving such problems, the Hough transform based method is inferior in relation to the incremental algorithm, and the incremental algorithm performed worse in relation to the statistical based method [1]. In similar examples, the new proposed method shows significant improvements.

The well-known EDCircles algorithm (available at: `http://ceng.anadolu.edu.tr/CV/EDCircles/demo.aspx`), proposed by Akinlar and Topal [3], has very good characteristics for the case of recognizing ellipses with clear edges, but it does not recognize ellipses with unclear or noisy edges.

[1] R.Grbić, D.Grahovac, R.Scitovski, *Pattern Recognition* **60**(2016)

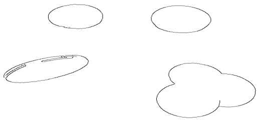
[2] R.Scitovski, T.Marošević, *Pattern Recognition Letters* **52**(2014)

[3] C.Akinlar, C.Topal, *Pattern Recognition* **46**(2013)

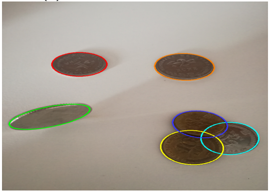
Recognizing ellipses in real-world images

(a) Original

(b) Canny filter

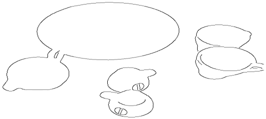


(c) Detection



(a) Original

(b) Canny filter



(c) Detection

[1] G.Bradski, *The opencv library, Dr. Dobb's Journal of Software Tools*, 2000

[2] Steven G. Johnson, *The NLopt nonlinear-optimization package*, 2011 `http://ab-initio.mit.edu/nlopt`

[3] D.R.Jones, C.D.Perttunen, B.E.Stuckman, *Journal of Optimization Theory and Applications* **79**(1993)

Public available program

http://cs.mathos.unios.hr/~pnikic/ells/