Least squares fitting problem for the power-law regression with a location parameter

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Content

1. Introduction

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Given the data (w_i, x_i, y_i), $\quad i = 1, \dots, n, \quad n > 3,$

$$x_1 < x_2 < \dots < x_n \tag{2}$$

$$y_i > 0, \quad i = 1, \dots, n, \qquad \text{and data weights} \;\; w_i > 0\,. \tag{3}$$

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The unknown parameters a, b and c, by minimizing the weighted sum of squares

$$F(a, b, c) = \sum_{i=1}^{n} w_i \big[a(x_i + b)^c - y_i\big]^2 \tag{4}$$

on set $D \subseteq \mathbb{R}^3$ (parameter space).

A point $(a^\star, b^\star, c^\star) \in D$ such that $F(a^\star, b^\star, c^\star) = \inf_{(a,b,c)\in D} F(a, b, c)$ is the best least squares (LS) estimator, if it exists.

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E. Demidenko: *Criteria for global minimum of sum of squares in nonlinear regression*, Comput. Statist. Data Anal. (2006)

D. Jukić: *A necessary and sufficient criteria for the existence of the least squares estimate for a 3-parametric exponential function*, Appl. Math. Comput. (2004)

2. Preliminaries

Let F be a lower bounded, real-valued continuous function defined on a noncompact set D in $\mathbb{R}^m$.
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- *extended boundary* of the set D:

$$\partial_\infty D = \begin{cases} \partial D, & \text{if } D \text{ is bounded} \\ \partial D \cup \{\infty\}, & \text{if } D \text{ is unbounded} \end{cases}$$

E. Demidenko: *Optimization and Regression*, Nauka, Moscow, 1989

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$$F^\star := \inf_{\substack{\{\boldsymbol{x}_k\} \subseteq D \\ \boldsymbol{x}_k \to \boldsymbol{x}_0 \in \partial_\infty D \setminus D}} \underline{\lim_{k}} F(\boldsymbol{x}_k), \tag{5}$$

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Remark:

$$F^\star := \inf_{\substack{\{\boldsymbol{x}_k\} \subseteq D \\ \boldsymbol{x}_k \to \boldsymbol{x}_0 \in \partial_\infty D \setminus D \\ \{F(\boldsymbol{x}_k)\} \text{ converges in } [-\infty, \infty]}} \lim_{k \to \infty} F(\boldsymbol{x}_k).$$

D. Jukić: *A necessary and sufficient criterion for the existence of the global minima of a continuous lower bounded function on a noncompact set*, J. Comput. Appl. Math. (2020)

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Theorem

Let F be a lower bounded real-valued continuous function defined on a noncompact set D in $\mathbb{R}^m$. Then:

(a) *The global minimizer of F exists if and only if there exists $\boldsymbol{x}_0 \in D$ such that*

$$F(\boldsymbol{x}_0) \leq F^\star.$$

(b) *If $F(\boldsymbol{x}_0) < F^\star$ for some $\boldsymbol{x}_0 \in D$, then the level set*

$$L(F(\boldsymbol{x}_0)) = \{\boldsymbol{x} \in D : F(\boldsymbol{x}) \leq F(\boldsymbol{x}_0)\}$$

is compact.

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the parameter space D

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\begin{aligned}
& D_+ := \{(a,b,c) \in \mathbb{R}^3 : a > 0, b \geq -x_1, c > 0\} \\
& \text{or} \\
& D_- := \{(a,b,c) \in \mathbb{R}^3 : a > 0, b \geq -x_1, c < 0\}.
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or (6)
$$D_- := \{(a,b,c) \in \mathbb{R}^3 : a > 0, b \geq -x_1, c < 0\}.$$

and $F : D \to \mathbb{R}$ is the weighted sum of squares given by (4)

$$F(a,b,c) = \sum_{i=1}^{n} w_i \left[a(x_i + b)^c - y_i \right]^2$$

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$$E^\star := \begin{cases} \inf\limits_{\alpha>0,\beta>0} \sum_{i=1}^n w_i(\alpha\, e^{\beta x_i} - y_i)^2, & \text{if } D = D_+ \\ \inf\limits_{\alpha>0,\beta<0} \sum_{i=1}^n w_i(\alpha\, e^{\beta x_i} - y_i)^2, & \text{if } D = D_-. \end{cases}$$

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The next theorem, together with Theorem 1, gives a necessary and sufficient criterion which guarantee the existence of the least squares estimate for the power-law regression with an unknown location parameter.

Theorem

Suppose that the data (w_i, x_i, y_i), $i = 1, \ldots, n$, $n > 3$, *satisfy the conditions (2) and (3), and let* D *be one of the sets in (6). Then the modified existence level* $F^\star$ *for the weighted sum of squares (4) reads*

$$F^\star = \begin{cases} \min\{E^\star, \sum_{i=2}^{n} w_i (y_i - \bar{y}_n)^2\}, & \text{if } y_1 \le \bar{y}_n \\ E^\star, & \text{if } y_1 > \bar{y}_n, \end{cases}$$

where $\bar{y}_n := \frac{\sum_{i=2}^{n} w_i y_i}{\sum_{i=2}^{n} w_i}$.

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D. Jukić, and T. Marošević: An existence level for the residual sum of squares of the power-law regression with an unknown location parameter, Mathematica Slovaca (2020), accepted for publication

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Let $\{(a_k, b_k, c_k)\} \subset D$ be any sequence such that

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and

$$F(a_k, b_k, c_k) = \sum_{i=1}^{n} [f(x_i; a_k, b_k, c_k) - y_i]^2 \text{ converges in } [0, \infty].$$

Denote

$$f_i^0 := \lim_{k\to\infty} f(x_i; a_k, b_k, c_k) = \lim_{k\to\infty} a_k(b_k + x_i)^{c_k}, \qquad i = 1, \ldots, n,$$

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$$0 \le f_1^0 \le f_2^0 \le \dots \le f_n^0 \le \infty.$$

Then only one of the following three cases can occur:

(a) $f_n^0 = \infty$,

(b) $f_n^0 = 0$, or

(c) $0 < f_n^0 < \infty$.

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(c) Case $0 < f_n^0 < \infty$. If $f_{n-1}^0 = 0$, then $f_i^0 = 0$ for all $i = 1, \ldots, n-1$, and therefore

$$\lim_{k\to\infty} F(a_k, b_k, c_k) \geq \sum_{i=1}^{n-1} w_i y_i^2. \tag{9}$$

So, suppose further that $f^0_{n-1} > 0$, and consider the following two subcases:

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$$\lim_{k\to\infty} F(a_k, b_k, c_k) \geq \begin{cases} \sum_{i=2}^{n} w_i(y_i - \bar{y}_n)^2, & \text{if } y_1 \leq \bar{y}_n \\ \sum_{i=1}^{n} w_i(y_i - \bar{y})^2, & \text{if } y_1 > \bar{y}_n \end{cases} \tag{10}$$

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From (7)-(11) we obtain:

$$F^\star = \begin{cases} \min\left\{\infty, \sum_{i=1}^{n} w_i y_i^2, \sum_{i=1}^{n-1} w_i y_i^2, \sum_{i=2}^{n} w_i (y_i - \bar{y}_n)^2, E^\star\right\}, & \text{if } y_1 \leq \bar{y}_n \\ \min\left\{\infty, \sum_{i=1}^{n} w_i y_i^2, \sum_{i=1}^{n-1} w_i y_i^2, \sum_{i=1}^{n} w_i (y_i - \bar{y})^2, E^\star\right\}, & \text{if } y_1 > \bar{y}_n. \end{cases}$$

... □

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The modified existence level [Jukić 2020]

$$F^\star := \inf_{\substack{\{\boldsymbol{x}_k\} \subseteq D \\ \boldsymbol{x}_k \to \boldsymbol{x}_0 \in \partial_\infty D \setminus D}} \varliminf_k F(\boldsymbol{x}_k)$$

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$$\overline{F} = \inf_{\substack{\{\mathbf{x}_k\}\subseteq D \\ \mathbf{x}_k \to \mathbf{x}_0 \in \partial_\infty D}} \underline{\lim_k} F(\mathbf{x}_k).$$

Since $\partial_\infty D \setminus D \subseteq \partial_\infty D$, we have

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For an exponential regression in some cases, the modified existence level $F^\star$ coincides with Demidenko's existence level $\overline{F}$.

KOI 2020

Certificate of Attendance

This is to certify that

Dragan Jukić

presented the paper

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Josip Arnerić
Conference Chair of the Organizing Committee of the
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