Skup kompleksnih brojeva - $\mathbb{C}$

Primjena kompleksnih brojeva

Primjenjivat ćemo poznati osnovni teorem algebre:

Osnovni teorem algebre. Svaki kompleksni polinom stupnja $n \geq 1$ ima točno n nultočaka u $\mathbb{C}$.

Zadatak 1. *Dokažite da je polinom*

$$P(x) = (\cos\alpha + x\sin\alpha)^n - \cos n\alpha - x\sin n\alpha$$

djeljiv s polinomom $Q(x) = x^2 + 1$.

Zadatak 2. *Dokažite sljedeće identitete:*

a) $x^{2n} - 1 = (x^2 - 1)\prod_{k=1}^{n-1}(x^2 - 2x\cos\frac{k\pi}{n} + 1)$,

b) $x^{2n+1} - 1 = (x - 1)\prod_{k=1}^{n}(x^2 - 2x\cos\frac{2k\pi}{2n+1} + 1)$,

c) $x^{2n} + 1 = \prod_{k=0}^{n-1}(x^2 - 2x\cos\frac{(2k+1)\pi}{2n} + 1)$,

Zadatak 3. *Dokažite sljedeće identitete:*

a) $\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{2n}\right) = \frac{\sqrt{n}}{2^{n-1}}$,

b) $\prod_{k=1}^{n-1}\cos\left(\frac{k\pi}{2n}\right) = \frac{\sqrt{n}}{2^{n-1}}$,

c) $\prod_{k=1}^{n-1}\operatorname{tg}\left(\frac{k\pi}{2n}\right) = 1$.

Zadatak 4. Dokažite identitet $\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$.

Zadatak 5. *Dokažite sljedeće identitete:*

a) $\prod_{k=1}^{n}\cos\left(\frac{k\pi}{2n+1}\right) = \frac{1}{2^n}$, *b)* $\displaystyle\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{2n+1}\right) = \frac{\sqrt{2n+1}}{2^n}$,

c) $\displaystyle\prod_{k=1}^{n-1} \operatorname{tg}\left(\frac{k\pi}{2n+1}\right) = \sqrt{2n+1}$.

Zadatak 6. *Izvedite formulu za sinus i kosinus trostrukog kuta.*

Zadatak 7. *Dokažite identitete:*

a) $1 + \cos\alpha + \cos 2\alpha + \cdots + \cos n\alpha = \dfrac{\sin\frac{n+1}{2}\alpha \cos\frac{n}{2}\alpha}{\sin\frac{\alpha}{2}}$

b) $\sin\alpha + \sin 2\alpha + \cdots + \sin n\alpha = \dfrac{\sin\frac{n+1}{2}\alpha \sin\frac{n}{2}\alpha}{\sin\frac{\alpha}{2}}$

Zadatak 8. *Dokažite identitete:*

a) $\sin^2 x + \sin^2(3x) + \cdots + \sin^2((2n-1)x) = \dfrac{n}{2} - \dfrac{\sin(4nx)}{4\sin(2x)}$,

b) $\cos^2 x + \cos^2(3x) + \cdots + \cos^2((2n-1)x) = \dfrac{n}{2} + \dfrac{\sin(4nx)}{4\sin(2x)}$,

Zadatak 9. *Dokažite identitete:*

a) $\cos^2 x + \cos^2(2x) + \cdots + \cos^2(nx) = \dfrac{n}{2} + \dfrac{\sin(nx)\cos((n+1)x)}{2\sin(x)}$,

b) $\sin^2 x + \sin^2(2x) + \cdots + \sin^2(nx) = \dfrac{n}{2} - \dfrac{\sin(nx)\cos((n+1)x)}{2\sin(x)}$.

Eksponencijalni prikaz kompleksnog broja

$$z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}$$

Zadatak 10. *Dokažite da za svaki $x \in \mathbb{R}$, $\varphi \in \mathbb{R}$ i $n \in \mathbb{N}$ vrijedi*

$$\sum_{k=0}^{n} \binom{n}{k} \sin(x + 2k\varphi) = 2^n \cos^n\varphi \sin(x + n\varphi).$$

Zadatak 11. *Izračunajte sljedeće sume*

$$A = \cos x + \binom{n}{1}\cos(2x) + \binom{n}{2}\cos(3x) + \cdots + \binom{n}{n}\cos((n+1)x),$$

$$B = \sin x + \binom{n}{1}\sin(2x) + \binom{n}{2}\sin(3x) + \cdots + \binom{n}{n}\sin((n+1)x).$$

Zadatak 12. *Izračunajte sljedeće sume*

$$C = \cos x - \binom{n}{1}\cos(2x) + \binom{n}{2}\cos(3x) + \cdots + (-1)^n\binom{n}{n}\cos((n+1)x),$$

$$B = \sin x - \binom{n}{1}\sin(2x) + \binom{n}{2}\sin(3x) + \cdots + (-1)^n\binom{n}{n}\sin((n+1)x).$$