2 Jednadžbe

2.4 Eksponencijalne i logaritamske jednadžbe

Zadatak 1. Riješite sljedeće jednadžbe:

a) $\log_4(x+6) - \log_4(x-1) = \log_4 10 - \log_4 2$

b) $\log(2x+5) - \log(3x-1) = \log(4x+8) - \log(6x-7)$
Rješenje: $x = \frac{-27}{4}$

c) $2 - \log_{\frac{1}{4}} x = \log_4 \left(x^2 + 10x + 6\right)$

d) $\frac{1}{2}\log_5(x-2) = 3\log_5 2 - \frac{3}{2}\log_5(x-2)$
Rješenje: $x = 2 + \sqrt{2}$

e) $3\sqrt{\log x} + 2\log\sqrt{\frac{1}{x}} = 2$

f) $\log(7x-9)^2 + \log(3x-4)^2 = 2$

g) $\log x^4 + \log(x+4)^4 = 4\log 3$
Rješenje: $x_1 = -3, x_2 = -1, x_{3,4} = -2 \pm \sqrt{7}$

Zadatak 2. Riješite sljedeće jednadžbe:

a) $\frac{1}{5-\log x} + \frac{4}{1+\log x} = 3$

b) $\frac{2\log x}{\log x - 1} - \log x = \frac{2}{\log x - 1}$
Rješenje: $x = 100$

c) $\log^2 x + 2\log(0.1x) = 1$

d) $\log(0.1x^2)\cdot\log\frac{10}{x} = -3$
Rješenje: $x_1 = 100, x_2 = \frac{1}{\sqrt{10}}$

e) $\log[3 + 2\log(x+1)] = 0$

f) $\log_4[4 - 2\log_5(4-x)] = \dfrac{1}{2}$
Rješenje: $x = -1$

g) $\log_3\{1 + \log_2[1 + \log_4(1 + \log_{\frac{1}{2}} x)]\} = 0$
Rješenje: $x = 1$

h) $\log_{16} x + \log_8 x + \log_2 x = \dfrac{19}{36}$

i) $\log_{125} x + \log_{25} x + \log_5 x = \dfrac{11}{6}$
Rješenje: $x = 5$

j) $\log_{\frac{1}{2}} x \cdot \log_2 x \cdot \log_4 x = 4$
Rješenje: $x = \frac{1}{4}$

k) $\log_3 x \cdot \log_9 x \cdot \log_{27} x \cdot \log_{81} x = \dfrac{2}{3}$
Rješenje: $x = 9$

Zadatak 3. Riješite sljedeće jednadžbe:

a) $\log_3(3^x - 8) = 2 - x$

b) $\log_5(5^x - 4) = 1 - x$
Rješenje: $x = 1$

c) $\log(2^x + 1) + x = x\log 5 + \log 6$

d) $\log_2(4^x + 4) = x + \log_2(2^{x+1} + 3)$
Rješenje: $x = 0$

e) $\log_5(5^x + 1) \cdot \log_5(5^{x+1} + 5) = 12$
Rješenje: $x_1 = \log_5 2, x_2 = \log_5(5^{-4} - 1)$

f) $x^{2\log x - 5} = 0.01$

g) $50^{\log x} \cdot 60^{\log x} = 400$
Rješenje: $x = 400^{\frac{1}{\log 3000}}$

2.5 Trigonometrijske jednadžbe

Napomena.

1. $\sin^2\alpha + \cos^2\alpha = 1$

2. $\sin 2\alpha = 2\sin\alpha\cos\alpha$
   $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

   $$\mathrm{tg}2\alpha = \frac{2\mathrm{tg}\alpha}{1-\mathrm{tg}^2\alpha}$$

   $$\mathrm{ctg}2\alpha = \frac{\mathrm{ctg}^2\alpha - 1}{2\mathrm{ctg}\alpha}$$

3. $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
   $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

   $$\mathrm{tg}(\alpha \pm \beta) = \frac{\mathrm{tg}\alpha \pm \mathrm{tg}\beta}{1 \mp \mathrm{tg}\alpha\mathrm{tg}\beta}$$

   $$\mathrm{ctg}(\alpha \pm \beta) = \frac{\mathrm{ctg}\alpha\mathrm{ctg}\beta \mp 1}{\mathrm{ctg}\beta \pm \mathrm{ctg}\alpha}$$

Napomena. Osnovne trigonometrijske jednadžbe su oblika

$$\sin x = p, \quad \cos x = p, \quad \mathrm{tg}\, x = r, \quad \mathrm{ctg}\, x = r, \ |p| \le 1.$$

Njihova rješenja su:

a) $\sin x = p \Longrightarrow x_{1k} = \arcsin p + 2k\pi,\ k \in \mathbb{Z}$ i $x_{2k} = \pi - \arcsin p + 2k\pi,\ k \in \mathbb{Z}$

b) $\cos x = p \Longrightarrow x_{1k} = \arccos p + 2k\pi,\ k \in \mathbb{Z}$ i $x_{2k} = -\arccos p + 2k\pi,\ k \in \mathbb{Z}$

c) $\mathrm{tg}\, x = r \Longrightarrow x_k = \mathrm{arctg}\, r + k\pi,\ k \in \mathbb{Z}$

d) $\mathrm{ctg}\, x = r \Longrightarrow x_k = \mathrm{arcctg}\, r + k\pi,\ k \in \mathbb{Z}$

Zadatak 4. Riješite sljedeće trigonometrijske jednadžbe:

a) $2\sin(x - \frac{\pi}{3}) = \sqrt{3}$

b) $\dfrac{\sqrt{3}}{\cos(3x-\frac{\pi}{3})}=2$
Rješenje: $x_1=\frac{\pi}{6}+\frac{2}{3}k\pi, x_2=\frac{\pi}{18}+\frac{2}{3}k\pi, k\in\mathbb{Z}$

c) $3\mathrm{tg}\,(x-\dfrac{\pi}{6})=-\sqrt{3}$
Rješenje: $x=k\pi, k\in\mathbb{Z}$

d) $\sin x+\sin 3x=0$

e) $\sin 5x=\cos 10x$

f) $\sin^3 2x-\sin 2x=0$
Rješenje: $x_1=\frac{k\pi}{2}, x_{2,3}=\pm\frac{\pi}{4}+\pi k, k\in\mathbb{Z}$

g) $\sin(\dfrac{\pi}{2}+x)+\mathrm{ctg}\,(2\pi-x)=0$

h) $2\sin^2 x+5\sin x+2=0$

i) $2\cos^2 x-3\sin x=0$
Rješenje: $x_1=\frac{\pi}{6}+2k\pi, x_2=\frac{5\pi}{6}+2k\pi, k\in\mathbb{Z}$

j) $\mathrm{tg}^2 x-(\sqrt{3}+1)\mathrm{tg}\,x+\sqrt{3}=0$
Rješenje: $x_1=-\frac{\pi}{3}+k\pi, x_2=\frac{\pi}{4}+k\pi, k\in\mathbb{Z}$

k) $2\sin^2 2x-\cos^2 2x=5\cos 2x$
Rješenje: $x_{1,2}=\pm\mathrm{arctg}(\frac{\sqrt{2}}{2})+\pi k, x_{3,4}=\pm\mathrm{arctg}(\sqrt{3})+\pi k, k\in\mathbb{Z}$

l) $11\mathrm{ctg}\,x-5\mathrm{tg}\,x=\dfrac{16}{\sin x}$

m) $\mathrm{tg}^2 x+\mathrm{ctg}^2 x=2$
Rješenje: $x_1=-\frac{\pi}{4}+\frac{\pi k}{2}, k\in\mathbb{Z}$

n) $4\cos^4 3x+8=11\sin^2 3x$
Rješenje: $x_{1,2}=\pm\frac{\pi}{9}+\frac{2}{3}k\pi, x_{3,4}=\pm\frac{2\pi}{9}+\frac{2}{3}k\pi, k\in\mathbb{Z}$

Napomena. **Homogena** trigonometrijska jednadžba ima oblik

$$a_n \sin^n x + a_{n-1} \sin^{n-1} x \cos x + \cdots + a_1 \sin x \cos^{n-1} x + a_0 \cos^n x = 0,$$

gdje su $a_0, a_1, \ldots, a_n$ realni brojevi, a rješava se dijeljenjem s $\cos^n x$.

Zadatak 5. Riješite sljedeće homogene trigonometrijske jednadžbe:

a) $2\sin^2 x - 5\sin x \cos x + 3\cos^2 x = 0$

b) $3\sin^2 x - 4\sin x \cos x + 5\cos^2 x = 2$
Rješenje: $x_1 = \frac{\pi}{4} + k\pi, x_2 = \mathrm{arctg}(-3) + k\pi, k \in \mathbb{Z}$

c) $\sin^2 x + 9\cos^2 x = 5\sin 2x$
Rješenje: $x_1 = \frac{\pi}{4} + k\pi, x_2 = \mathrm{arctg}(9) + \frac{k\pi}{2}, x_3 = \frac{\pi}{8} + \frac{k\pi}{2}, k \in \mathbb{Z}$

d) $8\sin^2 x + \sin x \cos x + \cos^2 x = 4$
Rješenje: $x_1 = -\frac{\pi}{4} + k\pi, x_2 = -2\mathrm{arctg}(3) + 2k\pi, x_3 = 2\mathrm{arctg}(\frac{1}{3}) + 2k\pi, k \in \mathbb{Z}$

e) $2\sin^4 2x - 9\sin^3 2x \cdot \cos 2x + 7\sin^2 2x \cdot \cos^2 2x = 0$

Napomena. **Linearne** trigonometrijske jednadžbe imaju oblik

$$a \sin x + b \cos x = c, \quad a, b, c \in \mathbb{R}, \ a, b \neq 0 \tag{1}$$

a mogu se rješavati na sljedeće načine:

1. Podijelimo li ovu jednadžbu s $\sqrt{a^2 + b^2}$ dobivamo:

$$\frac{a}{\sqrt{a^2+b^2}} \sin x + \frac{b}{\sqrt{a^2+b^2}} \cos x = \frac{c}{\sqrt{a^2+b^2}}.$$

Kako je

$$\left| \frac{a}{\sqrt{a^2+b^2}} \right|, \left| \frac{b}{\sqrt{a^2+b^2}} \right| \leq 1$$

postoji kut φ takav da vrijedi

$$\cos \varphi = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin \varphi = \frac{b}{\sqrt{a^2+b^2}}.$$

Gornja jednadžba prelazi u

$$\sin(x+\varphi) = \frac{c}{\sqrt{a^2+b^2}}.$$

2. Jednadžba (1) se može riješavati i uvođenjem univerzalne zamjene

$$t = \operatorname{tg}\frac{x}{2}.$$

(Tu se mora provjeriti je li $x = \pi + 2k\pi$, $k \in \mathbb{Z}$, (tj. x sa svojstvom $\cos\frac{x}{2} = 0$) rješenje jednadžbe.) Tada je

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

i jednadžba se svodi na kvadratnu po nepoznanici t.

Zadatak 6. Riješite sljedeće linearne jednadžbe:

a) $\sin x - 3\cos x = 1$

b) $\sqrt{3}\cos x - \sin x = 0$

c) $2\sin x - \cos x = 1$
Rješenje: $x_1 = \pi + 2k\pi, x_2 = \operatorname{arctg}(\frac{1}{2}), k \in \mathbb{Z}$

d) $8\sin x - 15\cos x = 17$
Rješenje: $x = \operatorname{arctg}(\frac{8}{17}) + (\frac{1}{2} + k)\pi, k \in \mathbb{Z}$

e) $3\sin x + 4\cos x = 6$
Rješenje: $x_{1,2} = \operatorname{arctg}(\frac{3\pm i\sqrt{11}}{10}) + k\pi, k \in \mathbb{Z}$

f) $\sin x + 7\cos x = -7$
Rješenje: $x_1 = \pi + 2k\pi, x_2 = -\operatorname{arctg}(7) + 2k\pi, k \in \mathbb{Z}$

g) $2\sqrt{3}\sin^2 x - \sin 2x = 2\sin x$