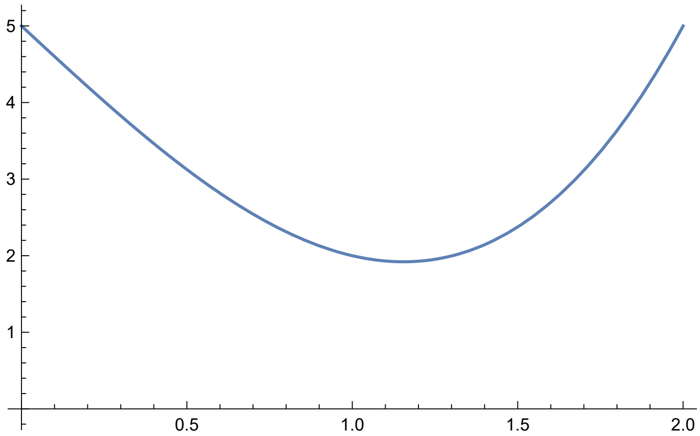
Vježbe 4. Klasične metode optimizacije

Newtonova metoda minimizacije (n=1)

Primjer 1.

```
f[x_] := x^3 - 4 x + 5
slf = Plot[f[x], {x, 0, 2}, AxesOrigin → {0, 0}]
```




```
x0 = 2;
eps = 0.0005;
k = 0;
Print["k=", k, "    xk= ", x0, "    f(xk)=", f[x0], "    f'(xk)=", f'[x0]];
While[Abs[f'[x0]] > eps,
 k++;
 x0 = x0 - f'[x0] / f''[x0];
 Print["k=", k, "    xk= ", x0, "    f(xk)=", f[x0], "    f'(xk)=", f'[x0]];
]
```

k=0 xk= 2 f(xk)=5 f'(xk)=8

k=1 xk= $\frac{4}{3}$ f(xk)=$\frac{55}{27}$ f'(xk)=$\frac{4}{3}$

k=2 xk= $\frac{7}{6}$ f(xk)=$\frac{415}{216}$ f'(xk)=$\frac{1}{12}$

k=3 xk= $\frac{97}{84}$ f(xk)=$\frac{1\,138\,465}{592\,704}$ f'(xk)=$\frac{1}{2352}$

```
FindMinimum[f[x], {x, 1}]
Solve[f'[x] == 0, x]
Solve[f'[x] == 0, x] // N
```

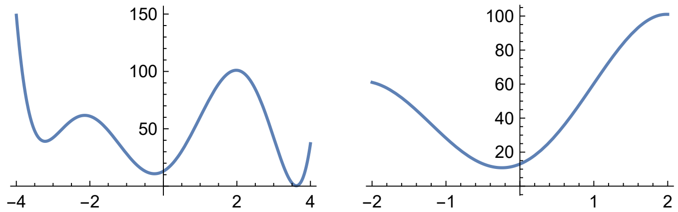
{1.9208, {x → 1.1547}}

$$\left\{\left\{x \to -\frac{2}{\sqrt{3}}\right\}, \left\{x \to \frac{2}{\sqrt{3}}\right\}\right\}$$

{{x → -1.1547}, {x → 1.1547}}

Primjer 2.

```
q[x_] := x^3 / 2 - 6 x + 2; u0 = 3; v0 = 4;
f[x_] := (x - u0) ^2 + (q[x] - v0) ^2
slf1 = Plot[f[x], {x, -4, 4}];
slf2 = Plot[f[x], {x, -2, 2}];
GraphicsGrid[{{slf1, slf2}}]
```



```
x0 = .5;
eps = 0.005;
k = 0;
Print["k=", k, "    xk= ", x0, "    f(xk)=", f[x0], "    f'(xk)=", f'[x0]];
While[Abs[f'[x0]] > eps,
 k++;
 x0 = x0 - f'[x0] / f''[x0];
 Print["k=", k, "    xk= ", x0, "    f(xk)=", f[x0], "    f'(xk)=", f'[x0]];
]
```

k=0   xk= 0.5   f(xk)=30.6289   f'(xk)=50.5469

k=1   xk= -0.501548   f(xk)=13.1561   f'(xk)=-17.6435

k=2   xk= -0.218716   f(xk)=10.8403   f'(xk)=1.77837

k=3   xk= -0.243011   f(xk)=10.8186   f'(xk)=0.00599434

k=4   xk= -0.243094   f(xk)=10.8186   f'(xk)=7.67311×10^{-8}

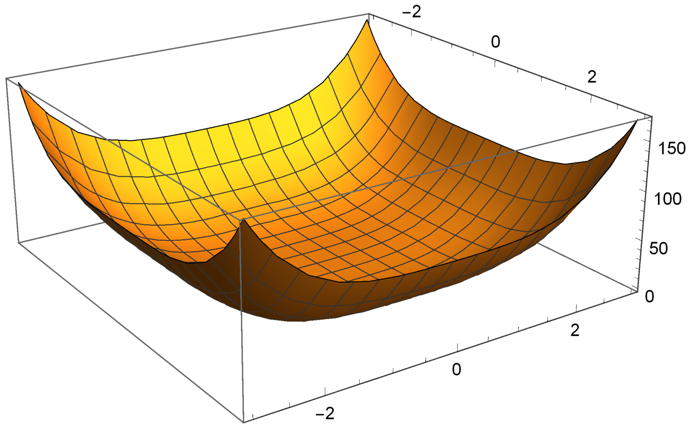
```
FindMinimum[f[x], {x, 0}]
```

{10.8186, {x → -0.243094}}

Newtonova metoda minimizacije (n=2)

Primjer 1.

```
F[x_, y_] := x^4 + y^4 + y^2;
Plot3D[F[x, y], {x, -3, 3}, {y, -3, 3}, PlotRange → All]
```




```
gradF = D[F[x, y], {{x, y}}]
hessianF = D[F[x, y], {{x, y}, 2}]
```

$\{4 x^3, 2 y + 4 y^3\}$

$\{\{12 x^2, 0\}, \{0, 2 + 12 y^2\}\}$

```
(*pokušati s razlicitim izborima pocetnih aproksimacija*)

x0 = 0.5; y0 = -0.5; (*ili x0=1/2; y0=-1/2;*)
x0 = 1 / 2; y0 = -1 / 2;
eps = 0.05;
k = 0;
Print["k=", k, "   ak= ", {x0, y0}, "   F(ak)=", F[x0, y0],
 "   grad(F)(ak)=", gradF /. {x → x0, y → y0}, "   hess(F)(ak)=",
 MatrixForm[hessianF /. {x → x0, y → y0}], "   ||grad(F)(ak)||=",
 Norm[gradF /. {x → x0, y → y0}], "   s(k-1)= ", s]
While[Norm[gradF /. {x → x0, y → y0}] > eps,
 k++;
 s = LinearSolve[hessianF /. {x → x0, y → y0}, -gradF /. {x → x0, y → y0}];
 {x0, y0} = {x0, y0} + s;
 Print["k=", k, "   ak= ", {x0, y0}, "   F(ak)=",
  F[x0, y0], "   grad(F)(ak)=", gradF /. {x → x0, y → y0},
  "   hess(F)(ak)=", MatrixForm[hessianF /. {x → x0, y → y0}],
  "   ||grad(F)(ak)||=", Norm[gradF /. {x → x0, y → y0}], "   s(k-1)= ", s]
]
```

k=0 ak= $\left\{\frac{1}{2}, -\frac{1}{2}\right\}$ F(ak) = $\frac{3}{8}$ grad(F)(ak) = $\left\{\frac{1}{2}, -\frac{3}{2}\right\}$ hess(F)(ak) =

$\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$ ||grad(F)(ak)|| = $\sqrt{\frac{5}{2}}$ s(k-1) = {-0.0740741, 0.025738}

k=1 ak= $\left\{\frac{1}{3}, -\frac{1}{5}\right\}$ F(ak) = $\frac{2731}{50\,625}$ grad(F)(ak) = $\left\{\frac{4}{27}, -\frac{54}{125}\right\}$

hess(F)(ak) = $\begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{62}{25} \end{pmatrix}$ ||grad(F)(ak)|| = $\frac{2\sqrt{593\,941}}{3375}$ s(k-1) = $\left\{-\frac{1}{6}, \frac{3}{10}\right\}$

k=2 ak= $\left\{\frac{2}{9}, -\frac{4}{155}\right\}$ F(ak) = $\frac{11\,758\,938\,016}{3\,787\,013\,300\,625}$

grad(F)(ak) = $\left\{\frac{32}{729}, -\frac{192\,456}{3\,723\,875}\right\}$ hess(F)(ak) = $\begin{pmatrix} \frac{16}{27} & 0 \\ 0 & \frac{48\,242}{24\,025} \end{pmatrix}$

||grad(F)(ak)|| = $\frac{8\sqrt{529\,441\,685\,477\,809}}{2\,714\,704\,875}$ s(k-1) = $\left\{-\frac{1}{9}, \frac{27}{155}\right\}$

k=3 ak= $\left\{\frac{4}{27}, -\frac{256}{3\,738\,755}\right\}$ F(ak) = $\frac{50\,020\,982\,707\,501\,965\,554\,783\,047\,936}{103\,839\,618\,526\,106\,572\,437\,596\,679\,650\,625}$

grad(F)(ak) = $\left\{\frac{256}{19\,683}, -\frac{7\,156\,884\,009\,521\,664}{52\,261\,397\,703\,350\,718\,875}\right\}$

hess(F)(ak) = $\begin{pmatrix} \frac{64}{243} & 0 \\ 0 & \frac{27\,956\,578\,686\,482}{13\,978\,288\,950\,025} \end{pmatrix}$ ||grad(F)(ak)|| =

$\frac{256\sqrt{2\,731\,556\,486\,241\,092\,457\,897\,874\,405\,920\,907\,373\,629}}{1\,028\,661\,090\,995\,052\,199\,616\,625}$ s(k-1) = $\left\{-\frac{2}{27}, \frac{96\,228}{3\,738\,755}\right\}$

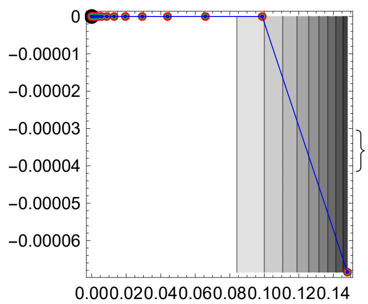
NMinimize[F[x, y], {x, y}]

$\{1.07058\times10^{-31}, \{x\to-1.80886\times10^{-8}, y\to0.\}\}$

<< Optimization`UnconstrainedProblems`

FindMinimumPlot[F[x, y], {{x, x0}, {y, y0}}, Method → Newton]

$\{\{8.25874\times10^{-31}, \{x\to3.01459\times10^{-8}, y\to-3.693028148721343\times10^{-601}\}\},$

{Steps → 38, Function → 39, Gradient → 39}, 

}

```
x0 = 1 / 2; y0 = -1 / 2;
eps = 0.05;
k = 0;
Print["k=", k, "    ak= ", {x0, y0}, "    F(ak)=", F[x0, y0],
 "    grad(F)(ak)=", gradF /. {x → x0, y → y0}, "    hess(F)(ak)=",
 MatrixForm[hessianF /. {x → x0, y → y0}], "    ||grad(F)(ak)||=",
 Norm[gradF /. {x → x0, y → y0}], "    s(k-1)= ", s]
While[Norm[gradF /. {x → x0, y → y0}] > eps,
 k++;
 s = LinearSolve[hessianF /. {x → x0, y → y0}, -gradF /. {x → x0, y → y0}];
 {x0, y0} = {x0, y0} + s;
 Print["k=", k, "    ak= ", {x0, y0}, "    F(ak)=",
  F[x0, y0], "    grad(F)(ak)=", gradF /. {x → x0, y → y0},
  "    hess(F)(ak)=", MatrixForm[hessianF /. {x → x0, y → y0}],
  "    ||grad(F)(ak)||=", Norm[gradF /. {x → x0, y → y0}], "    s(k-1)= ", s]
]
```

k=0   ak= $\left\{\frac{1}{2}, -\frac{1}{2}\right\}$ F(ak) = $\frac{3}{8}$ grad(F)(ak) = $\left\{\frac{1}{2}, -\frac{3}{2}\right\}$

hess(F)(ak) = $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$ ||grad(F)(ak)|| = $\sqrt{\frac{5}{2}}$ s(k-1) = $\left\{-\frac{2}{27}, \frac{96\,228}{3\,738\,755}\right\}$

k=1   ak= $\left\{\frac{1}{3}, -\frac{1}{5}\right\}$ F(ak) = $\frac{2731}{50\,625}$ grad(F)(ak) = $\left\{\frac{4}{27}, -\frac{54}{125}\right\}$

hess(F)(ak) = $\begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{62}{25} \end{pmatrix}$ ||grad(F)(ak)|| = $\frac{2\sqrt{593\,941}}{3375}$ s(k-1) = $\left\{-\frac{1}{6}, \frac{3}{10}\right\}$

k=2   ak= $\left\{\frac{2}{9}, -\frac{4}{155}\right\}$ F(ak) = $\frac{11\,758\,938\,016}{3\,787\,013\,300\,625}$

grad(F)(ak) = $\left\{\frac{32}{729}, -\frac{192\,456}{3\,723\,875}\right\}$ hess(F)(ak) = $\begin{pmatrix} \frac{16}{27} & 0 \\ 0 & \frac{48\,242}{24\,025} \end{pmatrix}$

||grad(F)(ak)|| = $\frac{8\sqrt{529\,441\,685\,477\,809}}{2\,714\,704\,875}$ s(k-1) = $\left\{-\frac{1}{9}, \frac{27}{155}\right\}$

k=3   ak= $\left\{\frac{4}{27}, -\frac{256}{3\,738\,755}\right\}$ F(ak) = $\frac{50\,020\,982\,707\,501\,965\,554\,783\,047\,936}{103\,839\,618\,526\,106\,572\,437\,596\,679\,650\,625}$

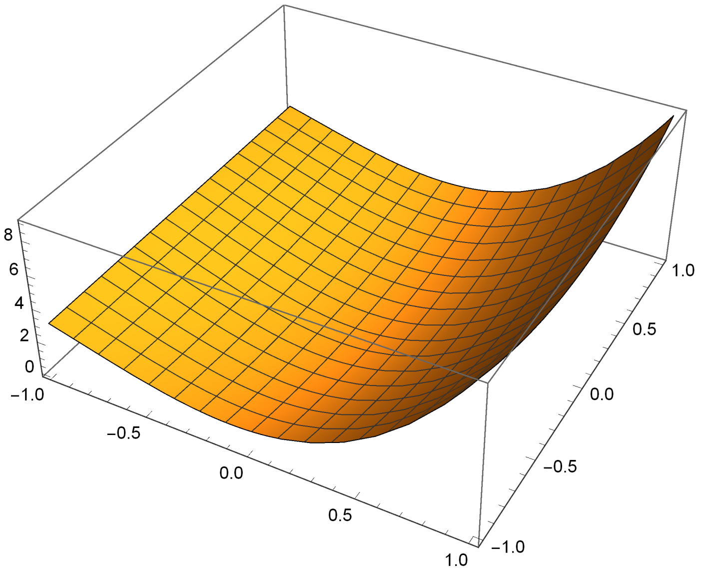
grad(F)(ak) = $\left\{\frac{256}{19\,683}, -\frac{7\,156\,884\,009\,521\,664}{52\,261\,397\,703\,350\,718\,875}\right\}$

hess(F)(ak) = $\begin{pmatrix} \frac{64}{243} & 0 \\ 0 & \frac{27\,956\,578\,686\,482}{13\,978\,288\,950\,025} \end{pmatrix}$ ||grad(F)(ak)|| =

$\frac{256\sqrt{2\,731\,556\,486\,241\,092\,457\,897\,874\,405\,920\,907\,373\,629}}{1\,028\,661\,090\,995\,052\,199\,616\,625}$ s(k-1) = $\left\{-\frac{2}{27}, \frac{96\,228}{3\,738\,755}\right\}$

Primjer 2.

```
F[x_, y_] := 2 x^3 + x y^2 + 5 x^2 + y^2;
Plot3D[F[x, y], {x, -1, 1}, {y, -1, 1}]
```




```
gradF = D[F[x, y], {{x, y}}]
hessianF = D[F[x, y], {{x, y}, 2}]
```

$\{10\,x + 6\,x^2 + y^2,\ 2\,y + 2\,x\,y\}$

$\{\{10 + 12\,x,\ 2\,y\},\ \{2\,y,\ 2 + 2\,x\}\}$

```
x0 = 1.; y0 = 1;
eps = 0.0005;
k = 0;
Print["k=", k, "    ak= ", {x0, y0}, "    F(ak)=",
 F[x0, y0], "    grad(F)(ak)=", -gradF /. {x → x0, y → y0},
 "    hess(F)(ak)=", MatrixForm[hessianF /. {x → x0, y → y0}]]
While[Norm[gradF /. {x → x0, y → y0}] > eps,
 k++;
 s = LinearSolve[hessianF /. {x → x0, y → y0}, -gradF /. {x → x0, y → y0}];
 {x0, y0} = {x0, y0} + s;
 Print["k=", k, "    ak= ", {x0, y0}, "    F(ak)=",
  F[x0, y0], "    grad(F)(ak)=", -gradF /. {x → x0, y → y0},
  "    hess(F)(ak)=", MatrixForm[hessianF /. {x → x0, y → y0}]]
]
```

k=0   ak= {1., 1}   F(ak)=9.   grad(F)(ak)={-17., -4.}   hess(F)(ak)= $\begin{pmatrix} 22. & 2 \\ 2 & 4. \end{pmatrix}$

k=1   ak= {0.285714, 0.357143}   F(ak)=0.618805   grad(F)(ak)= {-3.47449, -0.918367}   hess(F)(ak)= $\begin{pmatrix} 13.4286 & 0.714286 \\ 0.714286 & 2.57143 \end{pmatrix}$

k=2   ak= {0.0423772, 0.0675936}   F(ak)=0.0138939   grad(F)(ak)= {-0.439116, -0.140916}   hess(F)(ak)= $\begin{pmatrix} 10.5085 & 0.135187 \\ 0.135187 & 2.08475 \end{pmatrix}$

k=3   ak= {0.00142597, 0.00265551}   F(ak)=0.0000172346   grad(F)(ak)= {-0.014279, -0.0053186}   hess(F)(ak)= $\begin{pmatrix} 10.0171 & 0.00531102 \\ 0.00531102 & 2.00285 \end{pmatrix}$

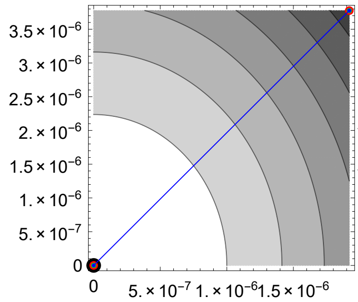
k=4   ak= $\{1.91992\times10^{-6}, 3.77621\times10^{-6}\}$ F(ak)= 3.26903×10^{-11} grad(F)(ak)= $\{-0.0000191993, -7.55243\times10^{-6}\}$ hess(F)(ak)= $\begin{pmatrix} 10. & 7.55241\times10^{-6} \\ 7.55241\times10^{-6} & 2. \end{pmatrix}$

```
NMinimize[F[x, y], {x, y}]
```

$\{1.21132\times10^{-44}, \{x\to3.69234\times10^{-23}, y\to7.27774\times10^{-23}\}\}$

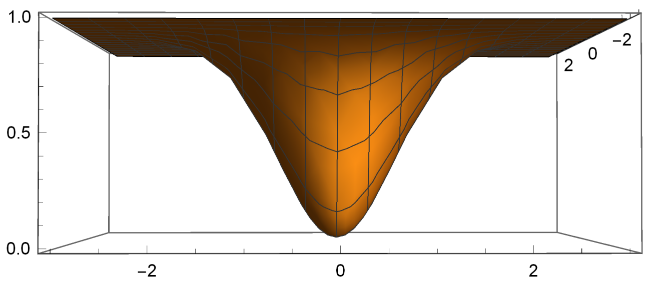
```
<< Optimization`UnconstrainedProblems`
FindMinimumPlot[F[x, y], {{x, x0}, {y, y0}}, Method → Newton]
```

$\{\{1.18724\times10^{-22}, \{x\to3.63763\times10^{-12}, y\to7.25001\times10^{-12}\}\},$

{Steps → 1, Function → 2, Gradient → 2},  }

Primjer 3.

```
F[x_, y_] := -Exp[-(x)^2 - (y)^2] + 1
Plot3D[F[x, y], {x, -3, 3}, {y, -3, 3}, PlotRange → All]
```



```
gradF = D[F[x, y], {{x, y}}]
hessianF = D[F[x, y], {{x, y}, 2}]
```

$\{2\, e^{-x^2-y^2}\, x,\ 2\, e^{-x^2-y^2}\, y\}$

$\{\{2\, e^{-x^2-y^2} - 4\, e^{-x^2-y^2}\, x^2,\ -4\, e^{-x^2-y^2}\, x\, y\},\ \{-4\, e^{-x^2-y^2}\, x\, y,\ 2\, e^{-x^2-y^2} - 4\, e^{-x^2-y^2}\, y^2\}\}$

```
(*pokušati s razlicitim izborima pocetnih aproksimacija*)
x0 = 2; y0 = -2;
eps = 0.0005;
k = 0;
Print["k=", k, "   ak= ", {x0, y0}, "   F(ak)=",
 F[x0, y0], "   grad(F)(ak)=", -gradF /. {x → x0, y → y0},
 "   hess(F)(ak)=", MatrixForm[hessianF /. {x → x0, y → y0}]]
While[Norm[gradF /. {x → x0, y → y0}] > eps,
 k++;
 s = LinearSolve[hessianF /. {x → x0, y → y0}, -gradF /. {x → x0, y → y0}];
 {x0, y0} = {x0, y0} + s;
 Print["k=", k, "   ak= ", {x0, y0}, "   F(ak)=",
  F[x0, y0], "   grad(F)(ak)=", -gradF /. {x → x0, y → y0},
  "   hess(F)(ak)=", MatrixForm[hessianF /. {x → x0, y → y0}]]
]
```

k=0   ak= {2, -2}   F(ak)= $1 - \frac{1}{e^8}$ grad(F)(ak)= $\{-\frac{4}{e^8},\ \frac{4}{e^8}\}$ hess(F)(ak)= $\begin{pmatrix} -\frac{14}{e^8} & \frac{16}{e^8} \\ \frac{16}{e^8} & -\frac{14}{e^8} \end{pmatrix}$

k=1   ak= $\{\frac{32}{15},\ -\frac{32}{15}\}$ F(ak)= $1 - \frac{1}{e^{2048/225}}$ grad(F)(ak)=
$\{-\frac{64}{15\, e^{2048/225}},\ \frac{64}{15\, e^{2048/225}}\}$ hess(F)(ak)= $\begin{pmatrix} -\frac{3646}{225\, e^{2048/225}} & \frac{4096}{225\, e^{2048/225}} \\ \frac{4096}{225\, e^{2048/225}} & -\frac{3646}{225\, e^{2048/225}} \end{pmatrix}$

k=2   ak= $\{\frac{131\,072}{58\,065},\ -\frac{131\,072}{58\,065}\}$ F(ak)= $1 - \frac{1}{e^{34\,359\,738\,368/3\,371\,544\,225}}$
grad(F)(ak)= $\{-\frac{262\,144}{58\,065\, e^{34\,359\,738\,368/3\,371\,544\,225}},\ \frac{262\,144}{58\,065\, e^{34\,359\,738\,368/3\,371\,544\,225}}\}$
hess(F)(ak)= $\begin{pmatrix} -\frac{61\,976\,388\,286}{3\,371\,544\,225\, e^{34\,359\,738\,368/3\,371\,544\,225}} & \frac{68\,719\,476\,736}{3\,371\,544\,225\, e^{34\,359\,738\,368/3\,371\,544\,225}} \\ \frac{68\,719\,476\,736}{3\,371\,544\,225\, e^{34\,359\,738\,368/3\,371\,544\,225}} & -\frac{61\,976\,388\,286}{3\,371\,544\,225\, e^{34\,359\,738\,368/3\,371\,544\,225}} \end{pmatrix}$

```
NMinimize[F[x, y], {x, y}]
```

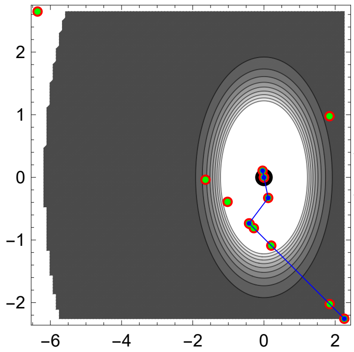
$\{0.,\ \{x \to -8.27181 \times 10^{-25},\ y \to 0.\}\}$

```
<< Optimization`UnconstrainedProblems`
```

```
FindMinimumPlot[F[x, y], {{x, x0}, {y, y0}}, Method → Newton]
```

$\{\{0., \{x \to 8.60268 \times 10^{-23}, y \to -2.31611 \times 10^{-22}\}\},$

{Steps → 6, Function → 16, Gradient → 16},

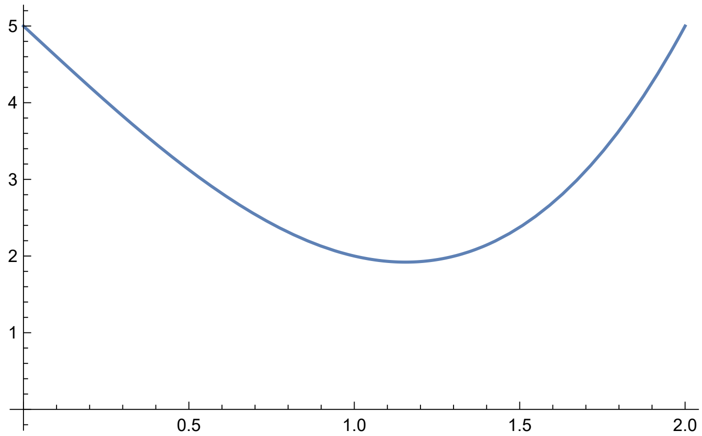


}

Newtonova metoda s izborom duljine koraka (n=1)

Primjer 1.

```
f[x_] := x^3 - 4 x + 5
slf = Plot[f[x], {x, 0, 2}, AxesOrigin → {0, 0}]
```



```
x0 = 0.5;
eps = 0.0005;
tau = 0.25; nu = 0.5; lam = 1;
k = 0;
Print["k=", k, "    xk= ", x0, "    f(xk)=",
  f[x0], "    f'(xk)=", f'[x0], "    f''(xk)=", f''[x0]];
While[Abs[f'[x0]] > eps,
 k++;
 s = -f'[x0] / f''[x0];
 lam = 1;
 Print[f[x0 + lam * s] - f[x0], " ? ", tau * lam * f'[x0] * s];
 While[f[x0 + lam * s] - f[x0] ≥ tau * lam * f'[x0] * s, lam = lam * nu ;
  Print[f[x0 + lam * s] - f[x0], " ? ", tau * lam * f'[x0] * s];];
 x0 = x0 + lam * s;
 Print["k=", k, "    xk= ", x0, "    f(xk)=", f[x0],
  "    f'(xk)=", f'[x0], "    f''(xk)=", f''[x0], "      lambda=", lam];
]
```

```
k=0    xk= 0.5    f(xk)=3.125    f'(xk)=-3.25    f''(xk)=3.

-0.489005 ? -0.880208

-1.16139 ? -0.440104

k=1    xk= 1.04167    f(xk)=1.96361    f'(xk)=-0.744792    f''(xk)=6.25      lambda=0.5

-0.0426849 ? -0.0221886

k=2    xk= 1.16083    f(xk)=1.92093    f'(xk)=0.0426021    f''(xk)=6.965      lambda=1

-0.000130519 ? -0.0000651449

k=3    xk= 1.15472    f(xk)=1.9208    f'(xk)=0.000112238    f''(xk)=6.9283      lambda=1
```

```
FindMinimum[f[x], {x, 1}]
Solve[f'[x] == 0, x]
Solve[f'[x] == 0, x] // N
```

{1.9208, {x → 1.1547}}

$$\left\{\left\{x \to -\frac{2}{\sqrt{3}}\right\}, \left\{x \to \frac{2}{\sqrt{3}}\right\}\right\}$$

{{x → -1.1547}, {x → 1.1547}}

```
x0 = 1 / 2;
eps = 0.0005;
tau = 1 / 4; nu = 1 / 2; lam = 1;
k = 0;
Print["k=", k, "    xk= ", x0, "    f(xk)=",
  f[x0], "    f'(xk)=", f'[x0], "    f''(xk)=", f''[x0]];
While[Abs[f'[x0]] > eps,
 k++;
 s = -f'[x0] / f''[x0];
 lam = 1;
 Print[f[x0 + lam * s] - f[x0], " ? ", tau * lam * f'[x0] * s];
 While[f[x0 + lam * s] - f[x0] ≥ tau * lam * f'[x0] * s, lam = lam * nu ;
  Print[f[x0 + lam * s] - f[x0], " ? ", tau * lam * f'[x0] * s;];
 x0 = x0 + lam * s;
 Print["k=", k, "    xk= ", x0, "    f(xk)=", f[x0],
  "    f'(xk)=", f'[x0], "    f''(xk)=", f''[x0], "       lambda=", lam];
]
```

k=0   xk= $\frac{1}{2}$ f(xk)=$\frac{25}{8}$ f'(xk)=$-\frac{13}{4}$ f''(xk)=3

$-\frac{845}{1728}$? $-\frac{169}{192}$

$-\frac{16\,055}{13\,824}$? $-\frac{169}{384}$

k=1   xk= $\frac{25}{24}$ f(xk)=$\frac{27\,145}{13\,824}$ f'(xk)=$-\frac{143}{192}$ f''(xk)=$\frac{25}{4}$ lambda=$\frac{1}{2}$

$-\frac{73\,759\,543}{1\,728\,000\,000}$? $-\frac{20\,449}{921\,600}$

k=2   xk= $\frac{1393}{1200}$ f(xk)=$\frac{3\,319\,365\,457}{1\,728\,000\,000}$ f'(xk)=$\frac{20\,449}{480\,000}$ f''(xk)=$\frac{1393}{200}$ lambda=1

$-\frac{4\,877\,078\,549\,571\,943}{37\,366\,900\,397\,568\,000\,000}$? $-\frac{418\,161\,601}{6\,418\,944\,000\,000}$

k=3   xk= $\frac{3\,860\,449}{3\,343\,200}$ f(xk)=$\frac{71\,774\,288\,670\,783\,058\,849}{37\,366\,900\,397\,568\,000\,000}$
    f'(xk)=$\frac{418\,161\,601}{3\,725\,662\,080\,000}$ f''(xk)=$\frac{3\,860\,449}{557\,200}$ lambda=1

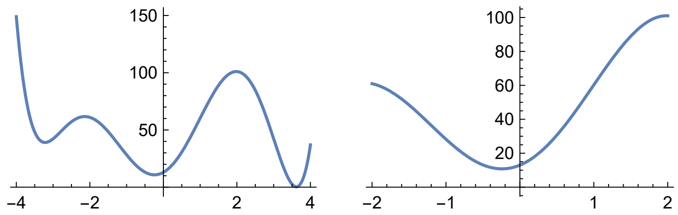
Bez izbora koraka :

```
x0 = 0.5;
eps = 0.0005;
k = 0;
Print["k=", k, "    xk= ", x0, "    f(xk)=", f[x0], "    f'(xk)=", f'[x0]];
While[Abs[f'[x0]] > eps,
 k++;
 x0 = x0 - f'[x0] / f''[x0];
 Print["k=", k, "    xk= ", x0, "    f(xk)=", f[x0], "    f'(xk)=", f'[x0]];
]
```

```
k=0    xk= 0.5    f(xk)=3.125    f'(xk)=-3.25

k=1    xk= 1.58333    f(xk)=2.636    f'(xk)=3.52083

k=2    xk= 1.21272    f(xk)=1.93265    f'(xk)=0.412064

k=3    xk= 1.15609    f(xk)=1.92081    f'(xk)=0.00962118

k=4    xk= 1.1547    f(xk)=1.9208    f'(xk)=5.77156×10^-6
```

Primjer 2.

```
q[x_] := x^3 / 2 - 6 x + 2; u0 = 3; v0 = 4;
f[x_] := (x - u0) ^2 + (q[x] - v0) ^2
slq = Plot[f[x], {x, -4, 4}];
slf = Plot[f[x], {x, -2, 2}];
GraphicsGrid[{{slq, slf}}]
```




```
x0 = .5;
eps = 0.005;
tau = 0.25; nu = 0.5; lam = 1;
k = 0;
Print["k=", k, "    xk= ", x0, "    f(xk)=",
  f[x0], "    f'(xk)=", f'[x0], "    f''(xk)=", f''[x0]];
While[Abs[f'[x0]] > eps,
 k++;
 s = -f'[x0] / f''[x0];
 lam = 1;
 While[f[x0 + lam * s] - f[x0] ≥ tau * lam * f'[x0] * s, lam = lam * nu ];
 x0 = x0 + lam * s;
 Print["k=", k, "    xk= ", x0, "    f(xk)=", f[x0],
  "    f'(xk)=", f'[x0], "    f''(xk)=", f''[x0], "      lambda=", lam];
]
```

```
k=0    xk= 0.5    f(xk)=30.6289    f'(xk)=50.5469    f''(xk)=50.4688

k=1    xk= -0.501548    f(xk)=13.1561
    f'(xk)=-17.6435    f''(xk)=62.3815      lambda=1

k=2    xk= -0.218716    f(xk)=10.8403    f'(xk)=1.77837    f''(xk)=73.1975      lambda=1

k=3    xk= -0.243011    f(xk)=10.8186
    f'(xk)=0.00599434    f''(xk)=72.6904      lambda=1

k=4    xk= -0.243094    f(xk)=10.8186
    f'(xk)=7.67311×10^-8    f''(xk)=72.6885      lambda=1
```

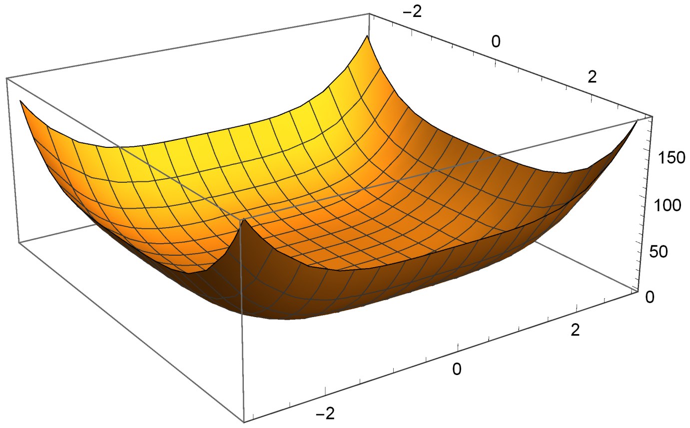
```
FindMinimum[f[x], {x, 0}]
```

```
{10.8186, {x → -0.243094}}
```

Newtonova metoda s izborom duljine koraka (n=2)

Primjer 1.

```
F[x_, y_] := Exp[x] + x^4 + y^4;
Plot3D[F[x, y], {x, -3, 3}, {y, -3, 3}, PlotRange → All]
```




```
gradF = D[F[x, y], {{x, y}}]
hessianF = D[F[x, y], {{x, y}, 2}]
```

$\{e^x + 4x^3, 4y^3\}$

$\{\{e^x + 12x^2, 0\}, \{0, 12y^2\}\}$

```
(*pokušati s razlicitim izborima pocetnih aproksimacija*)
```

```
x0 = 0.5; y0 = -0.5;
eps = 0.05;
tau = 0.25; nu = 0.5; lam = 1;
k = 0;
Print["k=", k, "   ak= ", {x0, y0}, "   F(ak)=", F[x0, y0],
 "   grad(F)(ak)=", gradF /. {x → x0, y → y0}, "   hess(F)(ak)=",
 MatrixForm[hessianF /. {x → x0, y → y0}], "   ||grad(F)(ak)||=",
 Norm[gradF /. {x → x0, y → y0}], "      lambda=", lam]
While[Norm[gradF /. {x → x0, y → y0}] > eps,
 k++;
 s = LinearSolve[hessianF /. {x → x0, y → y0}, -gradF /. {x → x0, y → y0}];
 lam = 1;
 Print[F[x0 + lam * s[[1]], y0 + lam * s[[2]]] - F[x0, y0],
  " ? ", tau * lam * ((gradF /. {x → x0, y → y0}).s)];
 While[F[x0 + lam * s[[1]], y0 + lam * s[[2]]] - F[x0, y0] ≥
   tau * lam * ((gradF /. {x → x0, y → y0}).s), lam = lam * nu ;
  Print[F[x0 + lam * s[[1]], y0 + lam * s[[2]]] - F[x0, y0],
   " ? ", tau * lam * ((gradF /. {x → x0, y → y0}).s)];];
 {x0, y0} = {x0, y0} + lam * s;
 Print["k=", k, "   ak= ", {x0, y0}, "   F(ak)=", F[x0, y0],
  "   grad(F)(ak)=", gradF /. {x → x0, y → y0}, "   hess(F)(ak)=",
  MatrixForm[hessianF /. {x → x0, y → y0}], "   ||grad(F)(ak)||=",
  Norm[gradF /. {x → x0, y → y0}], "   s(k-1)= ", s, "      lambda=", lam];
]
```

k=0   ak= {0.5, -0.5}   F(ak)=1.77372   grad(F)(ak)={2.14872, -0.5}
   hess(F)(ak)= $\begin{pmatrix} 4.64872 & 0 \\ 0 & 3. \end{pmatrix}$ ||grad(F)(ak)||=2.20613   lambda=1

-0.722868 ? -0.269128

k=1   ak= {0.0377823, -0.333333}   F(ak)=1.05085
   grad(F)(ak)={1.03872, -0.148148}   hess(F)(ak)= $\begin{pmatrix} 1.05564 & 0 \\ 0 & 1.33333 \end{pmatrix}$
   ||grad(F)(ak)||=1.04923   s(k-1)= {-0.462218, 0.166667}   lambda=1

0.141336 ? -0.259635

-0.367386 ? -0.129817

k=2   ak= {-0.454206, -0.277778}   F(ak)=0.683467   grad(F)(ak)=
 {0.260135, -0.0857339}   hess(F)(ak)= $\begin{pmatrix} 3.11059 & 0 \\ 0 & 0.925926 \end{pmatrix}$
   ||grad(F)(ak)||=0.273898   s(k-1)= {-0.983977, 0.111111}   lambda=0.5

-0.0146041 ? -0.00742326

k=3   ak= {-0.537835, -0.185185}   F(ak)=0.668862   grad(F)(ak)=
 {-0.0382991, -0.0254026}   hess(F)(ak)= $\begin{pmatrix} 4.05521 & 0 \\ 0 & 0.411523 \end{pmatrix}$
   ||grad(F)(ak)||=0.0459577   s(k-1)= {-0.0836287, 0.0925926}   lambda=1

```
NMinimize[F[x, y], {x, y}]
```

{0.667504, {x → -0.528252, y → -2.09783×10^{-8}}}

Bez izbora koraka :

```
x0 = 0.5; y0 = -0.5;
eps = 0.05;
tau = 0.25; nu = 0.5; lam = 1;
k = 0;
Print["k=", k, "   ak= ", {x0, y0}, "   F(ak)=", F[x0, y0],
 "   grad(F)(ak)=", gradF /. {x → x0, y → y0}, "   hess(F)(ak)=",
 MatrixForm[hessianF /. {x → x0, y → y0}], "   ||grad(F)(ak)||=",
 Norm[gradF /. {x → x0, y → y0}], "   s(k-1)= ", s, "     lambda=", lam]
While[Norm[gradF /. {x → x0, y → y0}] > eps,
 k++;
 s = LinearSolve[hessianF /. {x → x0, y → y0}, -gradF /. {x → x0, y → y0}];
 lam = 1;
 {x0, y0} = {x0, y0} + s;
 Print["k=", k, "   ak= ", {x0, y0}, "   F(ak)=", F[x0, y0],
  "   grad(F)(ak)=", gradF /. {x → x0, y → y0}, "   hess(F)(ak)=",
  MatrixForm[hessianF /. {x → x0, y → y0}], "   ||grad(F)(ak)||=",
  Norm[gradF /. {x → x0, y → y0}], "   s(k-1)= ", s, "     lambda=", lam];
]
```

k=0   ak= {0.5, -0.5}   F(ak)=1.77372   grad(F)(ak)= {2.14872, -0.5}   hess(F)(ak)= $\begin{pmatrix} 4.64872 & 0 \\ 0 & 3. \end{pmatrix}$ ||grad(F)(ak)||= 2.20613   s(k-1)= {-0.0836287, 0.0925926}   lambda=1

k=1   ak= {0.0377823, -0.333333}   F(ak)=1.05085   grad(F)(ak)={1.03872, -0.148148}   hess(F)(ak)= $\begin{pmatrix} 1.05564 & 0 \\ 0 & 1.33333 \end{pmatrix}$ ||grad(F)(ak)||=1.04923   s(k-1)= {-0.462218, 0.166667}   lambda=1

k=2   ak= {-0.946195, -0.222222}   F(ak)=1.19219   grad(F)(ak)= {-3.00024, -0.0438957}   hess(F)(ak)= $\begin{pmatrix} 11.1316 & 0 \\ 0 & 0.592593 \end{pmatrix}$ ||grad(F)(ak)||=3.00056   s(k-1)= {-0.983977, 0.111111}   lambda=1

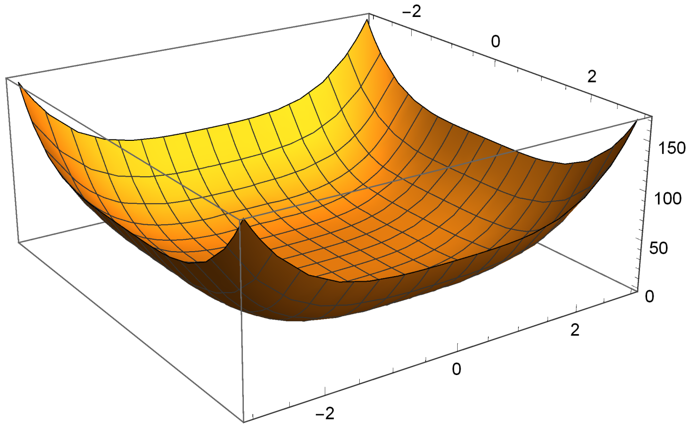
k=3   ak= {-0.676671, -0.148148}   F(ak)=0.718446   grad(F)(ak)= {-0.73104, -0.0130061}   hess(F)(ak)= $\begin{pmatrix} 6.00291 & 0 \\ 0 & 0.263374 \end{pmatrix}$ ||grad(F)(ak)||=0.731156   s(k-1)= {0.269524, 0.0740741}   lambda=1

k=4   ak= {-0.55489, -0.0987654}   F(ak)=0.669035   grad(F)(ak)= {-0.109274, -0.00385367}   hess(F)(ak)= $\begin{pmatrix} 4.26897 & 0 \\ 0 & 0.117055 \end{pmatrix}$ ||grad(F)(ak)||=0.109342   s(k-1)= {0.121781, 0.0493827}   lambda=1

k=5   ak= {-0.529293, -0.0658436}   F(ak)=0.667525   grad(F)(ak)= {-0.0041061, -0.00114183}   hess(F)(ak)= $\begin{pmatrix} 3.95083 & 0 \\ 0 & 0.0520246 \end{pmatrix}$ ||grad(F)(ak)||=0.00426191   s(k-1)= {0.0255973, 0.0329218}   lambda=1

Primjer 2.

```
F[x_, y_] := x^4 + y^4 + y^2;
Plot3D[F[x, y], {x, -3, 3}, {y, -3, 3}, PlotRange → All]
```




```
gradF = D[F[x, y], {{x, y}}]
hessianF = D[F[x, y], {{x, y}, 2}]
```

$\{4\,x^3,\ 2\,y + 4\,y^3\}$

$\{\{12\,x^2,\ 0\},\ \{0,\ 2 + 12\,y^2\}\}$

```
(*pokušati s razlicitim izborima pocetnih aproksimacija*)

x0 = 0.5; y0 = -0.5;
eps = 0.05;
tau = 0.25; nu = 0.5; lam = 1;
k = 0;
Print["k=", k, "    ak= ", {x0, y0}, "    F(ak)=", F[x0, y0],
 "    grad(F)(ak)=", gradF /. {x → x0, y → y0}, "    hess(F)(ak)=",
 MatrixForm[hessianF /. {x → x0, y → y0}], "    ||grad(F)(ak)||=",
 Norm[gradF /. {x → x0, y → y0}], "    s(k-1)= ", s, "      lambda=", lam]
While[Norm[gradF /. {x → x0, y → y0}] > eps,
 k++;
 s = LinearSolve[hessianF /. {x → x0, y → y0}, -gradF /. {x → x0, y → y0}];
 lam = 1;
 While[F[x0 + lam * s[[1]], y0 + lam * s[[2]]] - F[x0, y0] ≥
   tau * lam * ((gradF /. {x → x0, y → y0}).s), lam = lam * nu ];
 {x0, y0} = {x0, y0} + lam * s;
 Print["k=", k, "    ak= ", {x0, y0}, "    F(ak)=", F[x0, y0],
  "    grad(F)(ak)=", gradF /. {x → x0, y → y0}, "    hess(F)(ak)=",
  MatrixForm[hessianF /. {x → x0, y → y0}], "    ||grad(F)(ak)||=",
  Norm[gradF /. {x → x0, y → y0}], "    s(k-1)= ", s, "      lambda=", lam];
]
```

```
k=0   ak= {0.5, -0.5}   F(ak)=0.375   grad(F)(ak)={0.5, -1.5}   hess(F)(ak)=( 3.  0
                                                                             0   5. )
    ||grad(F)(ak)||=1.58114   s(k-1)= {0.0255973, 0.0329218}    lambda=1
k=1   ak= {0.333333, -0.2}   F(ak)=0.0539457
    grad(F)(ak)={0.148148, -0.432}   hess(F)(ak)=( 1.33333   0
                                                   0        2.48 )
    ||grad(F)(ak)||=0.456697   s(k-1)= {-0.166667, 0.3}    lambda=1
k=2   ak= {0.222222, -0.0258065}   F(ak)=0.00310507   grad(F)(ak)=
 {0.0438957, -0.0516816}   hess(F)(ak)=( 0.592593    0
                                         0        2.00799 )
    ||grad(F)(ak)||=0.0678073   s(k-1)= {-0.111111, 0.174194}    lambda=1
k=3   ak= {0.148148, -0.000068472}   F(ak)=0.000481714
    grad(F)(ak)={0.0130061, -0.000136944}   hess(F)(ak)=( 0.263374  0
                                                          0        2. )
    ||grad(F)(ak)||=0.0130069   s(k-1)= {-0.0740741, 0.025738}    lambda=1
```

```
NMinimize[F[x, y], {x, y}]
```

$\{1.07058\times10^{-31}, \{x \to -1.80886\times10^{-8}, y \to 0.\}\}$

```
x0 = 1 / 2; y0 = -1 / 2;
eps = 0.05;
k = 0;
tau = 0.25; nu = 0.5; lam = 1;
k = 0;
Print["k=", k, "   ak= ", {x0, y0}, "   F(ak)=", F[x0, y0],
 "   grad(F)(ak)=", gradF /. {x → x0, y → y0}, "   hess(F)(ak)=",
 MatrixForm[hessianF /. {x → x0, y → y0}], "   ||grad(F)(ak)||=",
 Norm[gradF /. {x → x0, y → y0}], "   s(k-1)= ", s, "     lambda=", lam]
While[Norm[gradF /. {x → x0, y → y0}] > eps,
 k++;
 s = LinearSolve[hessianF /. {x → x0, y → y0}, -gradF /. {x → x0, y → y0}];
 lam = 1;
 While[F[x0 + lam * s[[1]], y0 + lam * s[[2]]] - F[x0, y0] ≥
   tau * lam * ((gradF /. {x → x0, y → y0}).s), lam = lam * nu ];
 {x0, y0} = {x0, y0} + lam * s;
 Print["k=", k, "   ak= ", {x0, y0}, "   F(ak)=", F[x0, y0],
  "   grad(F)(ak)=", gradF /. {x → x0, y → y0}, "   hess(F)(ak)=",
  MatrixForm[hessianF /. {x → x0, y → y0}], "   ||grad(F)(ak)||=",
  Norm[gradF /. {x → x0, y → y0}], "   s(k-1)= ", s, "     lambda=", lam];
]
```

k=0 ak= $\left\{\frac{1}{2}, -\frac{1}{2}\right\}$ F(ak) = $\frac{3}{8}$ grad(F)(ak) = $\left\{\frac{1}{2}, -\frac{3}{2}\right\}$ hess(F)(ak) = $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$

||grad(F)(ak)|| = $\sqrt{\frac{5}{2}}$ s(k-1) = {-0.0740741, 0.025738} lambda=1

k=1 ak= $\left\{\frac{1}{3}, -\frac{1}{5}\right\}$ F(ak) = $\frac{2731}{50\,625}$ grad(F)(ak) = $\left\{\frac{4}{27}, -\frac{54}{125}\right\}$ hess(F)(ak) =

$\begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{62}{25} \end{pmatrix}$ ||grad(F)(ak)|| = $\frac{2\sqrt{593\,941}}{3375}$ s(k-1) = $\left\{-\frac{1}{6}, \frac{3}{10}\right\}$ lambda=1

k=2 ak= $\left\{\frac{2}{9}, -\frac{4}{155}\right\}$ F(ak) = $\frac{11\,758\,938\,016}{3\,787\,013\,300\,625}$ grad(F)(ak) =

$\left\{\frac{32}{729}, -\frac{192\,456}{3\,723\,875}\right\}$ hess(F)(ak) = $\begin{pmatrix} \frac{16}{27} & 0 \\ 0 & \frac{48\,242}{24\,025} \end{pmatrix}$ ||grad(F)(ak)|| =

$\frac{8\sqrt{529\,441\,685\,477\,809}}{2\,714\,704\,875}$ s(k-1) = $\left\{-\frac{1}{9}, \frac{27}{155}\right\}$ lambda=1

k=3 ak= $\left\{\frac{4}{27}, -\frac{256}{3\,738\,755}\right\}$ F(ak) = $\frac{50\,020\,982\,707\,501\,965\,554\,783\,047\,936}{103\,839\,618\,526\,106\,572\,437\,596\,679\,650\,625}$

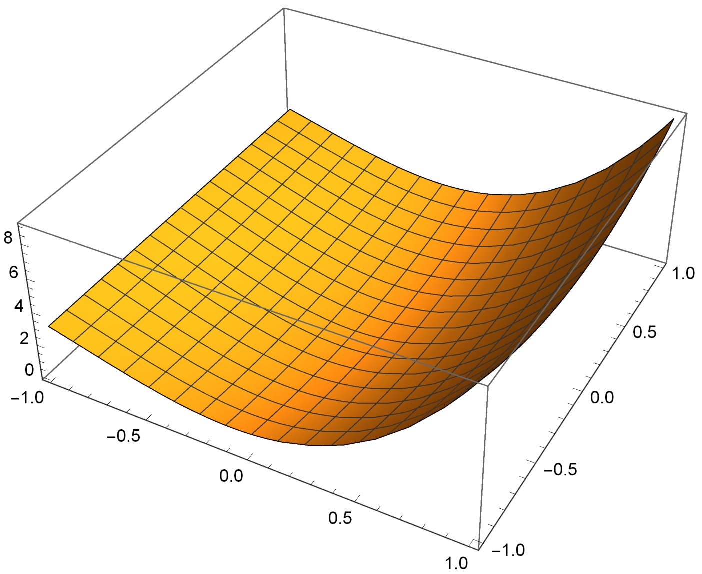
grad(F)(ak) = $\left\{\frac{256}{19\,683}, -\frac{7\,156\,884\,009\,521\,664}{52\,261\,397\,703\,350\,718\,875}\right\}$ hess(F)(ak) = $\begin{pmatrix} \frac{64}{243} & 0 \\ 0 & \frac{27\,956\,578\,686\,482}{13\,978\,288\,950\,025} \end{pmatrix}$

||grad(F)(ak)|| = $\frac{256\sqrt{2\,731\,556\,486\,241\,092\,457\,897\,874\,405\,920\,907\,373\,629}}{1\,028\,661\,090\,995\,052\,199\,616\,625}$

s(k-1) = $\left\{-\frac{2}{27}, \frac{96\,228}{3\,738\,755}\right\}$ lambda=1

Primjer 3.

```
F[x_, y_] := 2 x^3 + x y^2 + 5 x^2 + y^2;
Plot3D[F[x, y], {x, -1, 1}, {y, -1, 1}]
```



```
gradF = D[F[x, y], {{x, y}}]
hessianF = D[F[x, y], {{x, y}, 2}]
```

$\{10\,x + 6\,x^2 + y^2,\ 2\,y + 2\,x\,y\}$

$\{\{10 + 12\,x,\ 2\,y\},\ \{2\,y,\ 2 + 2\,x\}\}$

```
x0 = 1.; y0 = 1;
eps = 0.0005;
tau = 0.25; nu = 0.5; lam = 1;
k = 0;
Print["k=", k, "   ak= ", {x0, y0}, "   F(ak)=",
 F[x0, y0], "   grad(F)(ak)=", -gradF /. {x → x0, y → y0},
 "   hess(F)(ak)=", MatrixForm[hessianF /. {x → x0, y → y0}]]
While[Norm[gradF /. {x → x0, y → y0}] > eps,
 k++;
 s = LinearSolve[hessianF /. {x → x0, y → y0}, -gradF /. {x → x0, y → y0}];
 lam = 1;
 While[F[x0 + lam * s[[1]], y0 + lam * s[[2]]] - F[x0, y0] ≥
   tau * lam * ((gradF /. {x → x0, y → y0}).s), lam = lam * nu ];
 {x0, y0} = {x0, y0} + lam * s;
 Print["k=", k, "   ak= ", {x0, y0}, "   F(ak)=", F[x0, y0],
  "   grad(F)(ak)=", -gradF /. {x → x0, y → y0}, "   hess(F)(ak)=",
  MatrixForm[hessianF /. {x → x0, y → y0}], "     lambda=", lam];
]
```

k=0   ak= {1., 1}   F(ak)=9.   grad(F)(ak)={-17., -4.}   hess(F)(ak)= $\begin{pmatrix} 22. & 2 \\ 2 & 4. \end{pmatrix}$

k=1   ak= {0.285714, 0.357143}   F(ak)=0.618805   grad(F)(ak)= {-3.47449, -0.918367}   hess(F)(ak)= $\begin{pmatrix} 13.4286 & 0.714286 \\ 0.714286 & 2.57143 \end{pmatrix}$ lambda=1

k=2   ak= {0.0423772, 0.0675936}   F(ak)=0.0138939   grad(F)(ak)= {-0.439116, -0.140916}   hess(F)(ak)= $\begin{pmatrix} 10.5085 & 0.135187 \\ 0.135187 & 2.08475 \end{pmatrix}$ lambda=1

k=3   ak= {0.00142597, 0.00265551}   F(ak)=0.0000172346   grad(F)(ak)= {-0.014279, -0.0053186}   hess(F)(ak)= $\begin{pmatrix} 10.0171 & 0.00531102 \\ 0.00531102 & 2.00285 \end{pmatrix}$ lambda=1

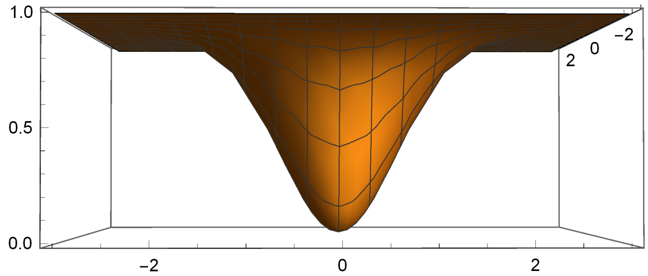
k=4   ak= $\{1.91992\times10^{-6},\ 3.77621\times10^{-6}\}$ F(ak)= 3.26903×10^{-11} grad(F)(ak)= $\{-0.0000191993,\ -7.55243\times10^{-6}\}$ hess(F)(ak)= $\begin{pmatrix} 10. & 7.55241\times10^{-6} \\ 7.55241\times10^{-6} & 2. \end{pmatrix}$ lambda=1

```
NMinimize[F[x, y], {x, y}]
```

$\{1.21132\times10^{-44},\ \{x \to 3.69234\times10^{-23},\ y \to 7.27774\times10^{-23}\}\}$

Primjer 4.

```
F[x_, y_] := -Exp[-(x)^2 - (y)^2] + 1
Plot3D[F[x, y], {x, -3, 3}, {y, -3, 3}, PlotRange → All]
```




```
gradF = D[F[x, y], {{x, y}}]
hessianF = D[F[x, y], {{x, y}, 2}]
```

$\{2 e^{-x^2-y^2} x, 2 e^{-x^2-y^2} y\}$

$\{\{2 e^{-x^2-y^2} - 4 e^{-x^2-y^2} x^2, -4 e^{-x^2-y^2} x y\}, \{-4 e^{-x^2-y^2} x y, 2 e^{-x^2-y^2} - 4 e^{-x^2-y^2} y^2\}\}$

```
(*pokušati s razlicitim izborima pocetnih aproksimacija*)
```

```
x0 = 0.3; y0 = 0.3;
eps = 0.0005;
tau = 0.25; nu = 0.5; lam = 1;
k = 0;
Print["k=", k, "   ak= ", {x0, y0}, "   F(ak)=",
 F[x0, y0], "   grad(F)(ak)=", -gradF /. {x → x0, y → y0},
 "   hess(F)(ak)=", MatrixForm[hessianF /. {x → x0, y → y0}]]
While[Norm[gradF /. {x → x0, y → y0}] > eps,
 k++;
 s = LinearSolve[hessianF /. {x → x0, y → y0}, -gradF /. {x → x0, y → y0}];
 lam = 1;
 While[F[x0 + lam * s[[1]], y0 + lam * s[[2]]] - F[x0, y0] ≥
   tau * lam * ((gradF /. {x → x0, y → y0}).s), lam = lam * nu ];
 {x0, y0} = {x0, y0} + lam * s;
 Print["k=", k, "   ak= ", {x0, y0}, "   F(ak)=", F[x0, y0],
  "   grad(F)(ak)=", -gradF /. {x → x0, y → y0}, "   hess(F)(ak)=",
  MatrixForm[hessianF /. {x → x0, y → y0}], "     lambda=", lam];
]
```

k=0 ak= {0.3, 0.3} F(ak)=0.16473 grad(F)(ak)=
{-0.501162, -0.501162} hess(F)(ak)= $\begin{pmatrix} 1.36984 & -0.300697 \\ -0.300697 & 1.36984 \end{pmatrix}$

k=1 ak= {0.065625, 0.065625} F(ak)=0.00857629 grad(F)(ak)=
{-0.130124, -0.130124} hess(F)(ak)= $\begin{pmatrix} 1.96577 & -0.0170788 \\ -0.0170788 & 1.96577 \end{pmatrix}$ lambda=0.5

k=2 ak= {-0.00115031, -0.00115031} F(ak)= 2.64642×10^{-6} grad(F)(ak)=
{0.00230061, 0.00230061} hess(F)(ak)= $\begin{pmatrix} 1.99999 & -5.29283\times 10^{-6} \\ -5.29283\times 10^{-6} & 1.99999 \end{pmatrix}$ lambda=1

k=3 ak= $\{6.08844\times 10^{-9}, 6.08844\times 10^{-9}\}$ F(ak)=
1.11022×10^{-16} grad(F)(ak)= $\{-1.21769\times 10^{-8}, -1.21769\times 10^{-8}\}$
hess(F)(ak)= $\begin{pmatrix} 2. & -1.48276\times 10^{-16} \\ -1.48276\times 10^{-16} & 2. \end{pmatrix}$ lambda=1

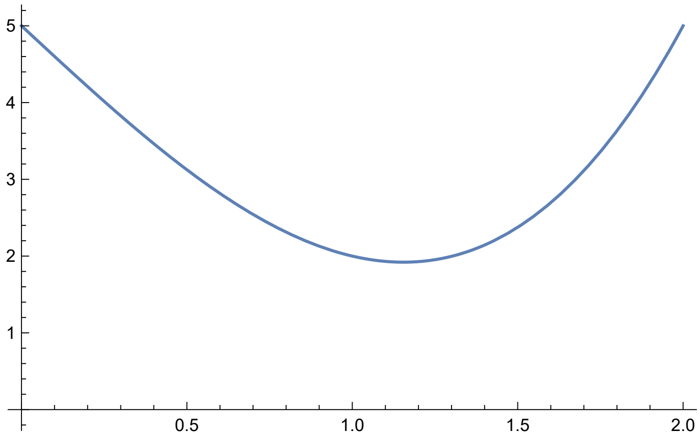
```
NMinimize[F[x, y], {x, y}]
```

$\{0., \{x \to -8.27181\times 10^{-25}, y \to 0.\}\}$

Kvadratne interpolacijske metode

Primjer 1.

```
f[x_] := x^3 - 4 x + 5
slf = Plot[f[x], {x, 0, 2}, AxesOrigin → {0, 0}]
```



Metoda dvije točke - Metoda I

```
x0 = 0; x1 = 2;
eps = 0.005;
k = 2;
Print["k=", 0, "    xk= ", x0, "    f[xk]= ", f[x0], "    f'[x2]= ", f'[x0]]
Print["k=", 1, "    xk= ", x1, "    f[xk]= ", f[x1], "    f'[x2]= ", f'[x1]]
While[True,
 x2 = x0 - (x1 - x0) * f'[x0] / (f'[x1] - f'[x0]);
 Print["k=", k, "    xk= ", x2, "    f[xk]= ", f[x2], "    f'[x2]= ", f'[x2]];
 If[Abs[f'[x2]] < eps, Break[]];
 x0 = x1;
 x1 = x2;
 k++;
]
```

k=0 xk= 0 f[xk]= 5 f'[x2]= -4

k=1 xk= 2 f[xk]= 5 f'[x2]= 8

k=2 xk= $\frac{2}{3}$ f[xk]= $\frac{71}{27}$ f'[x2]= $-\frac{8}{3}$

k=3 xk= 1 f[xk]= 2 f'[x2]= -1

k=4 xk= $\frac{6}{5}$ f[xk]= $\frac{241}{125}$ f'[x2]= $\frac{8}{25}$

k=5 xk= $\frac{38}{33}$ f[xk]= $\frac{69029}{35937}$ f'[x2]= $-\frac{8}{363}$

k=6 xk= $\frac{112}{97}$ f[xk]= $\frac{1753061}{912673}$ f'[x2]= $-\frac{4}{9409}$

```
FindMinimum[f[x], {x, 1}]
Solve[f'[x] == 0, x]
Solve[f'[x] == 0, x] // N
```

{1.9208, {x → 1.1547}}

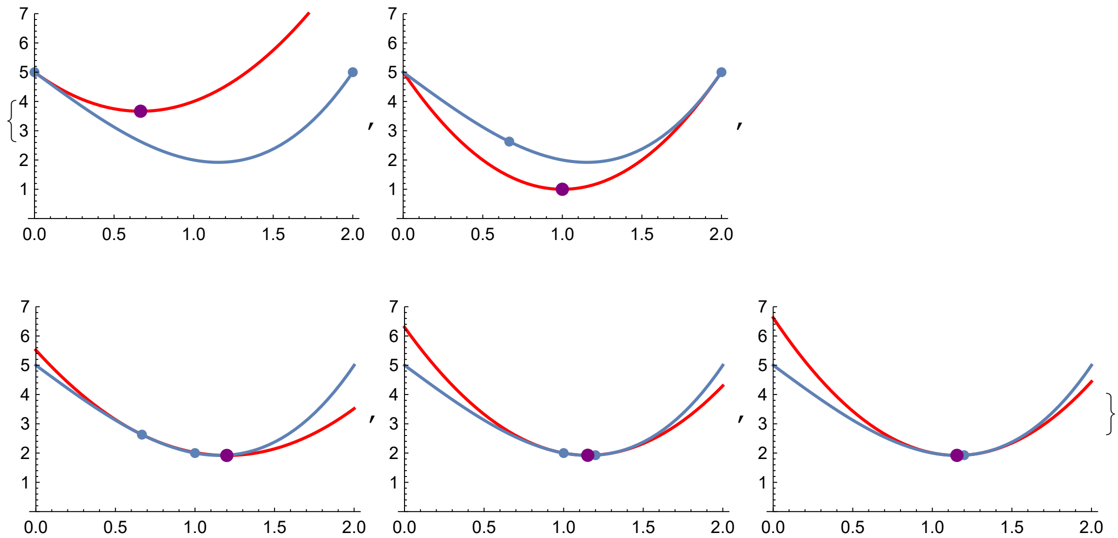
$$\left\{\left\{x \to -\frac{2}{\sqrt{3}}\right\}, \left\{x \to \frac{2}{\sqrt{3}}\right\}\right\}$$

{{x → -1.1547}, {x → 1.1547}}

Ilustracija :

```
x0 = 0.; x1 = 2.;
eps = 0.005;
k = 2;
slpar = {};
Print["k=", 0, "    xk= ", x0, "    f[xk]= ", f[x0], "    f'[x2]= ", f'[x0]]
Print["k=", 1, "    xk= ", x1, "    f[xk]= ", f[x1], "    f'[x2]= ", f'[x1]]
While[True,
 {al, be, ga} = {alfa, beta, gama} /.
   Solve[alfa * x0^2 + beta * x0 + gama == f[x0] && 2 * alfa * x0 + beta == f'[x0] &&
      2 * alfa * x1 + beta == f'[x1], {alfa, beta, gama}][[1]];
 x2 = x0 - (x1 - x0) * f'[x0] / (f'[x1] - f'[x0]);
 AppendTo[slpar, Show[{Plot[al * x^2 + be * x + ga, {x, 0, 2},
      PlotStyle → Red, PlotRange → {0, 7}, AxesOrigin → {0, 0}], slf,
    ListPlot[{{x0, f[x0]}, {x1, f[x1]}}, PlotStyle → PointSize[0.03]], ListPlot[
      {{x2, al * x2^2 + be * x2 + ga}}, PlotStyle → {Purple, PointSize[0.04]}]}]];
 Print["k=", k, "    xk= ", x2, "    f[xk]= ", f[x2], "    f'[x2]= ", f'[x2]];
 If[Abs[f'[x2]] < eps, Break[]];
 x0 = x1;
 x1 = x2;
 k++;
]
slpar
```

```
k=0    xk= 0.    f[xk]= 5.    f'[x2]= -4.

k=1    xk= 2.    f[xk]= 5.    f'[x2]= 8.

k=2    xk= 0.666667    f[xk]= 2.62963    f'[x2]= -2.66667

k=3    xk= 1.    f[xk]= 2.    f'[x2]= -1.

k=4    xk= 1.2    f[xk]= 1.928    f'[x2]= 0.32

k=5    xk= 1.15152    f[xk]= 1.92083    f'[x2]= -0.0220386

k=6    xk= 1.15464    f[xk]= 1.9208    f'[x2]= -0.000425125
```



Metoda dvije točke - Metoda II

```
x0 = 0; x1 = 2;
eps = 0.005;
k = 2;
Print["k=", 0, "    xk= ", x0, "    f[xk]= ", f[x0], "    f'[x2]= ", f'[x0]]
Print["k=", 1, "    xk= ", x1, "    f[xk]= ", f[x1], "    f'[x2]= ", f'[x1]]
While[True,
 x2 = x0 - 1 / 2 * ((x0 - x1) * f'[x0]) / (f'[x0] - (f[x1] - f[x0]) / (x1 - x0));
 Print["k=", k, "    xk= ", x2, "    f[xk]= ", f[x2], "    f'[x2]= ", f'[x2]];
 If[Abs[f'[x2]] < eps, Break[]];
 x0 = x1;
 x1 = x2;
 k++;
]
```

k=0 xk= 0 f[xk]= 5 f'[x2]= -4

k=1 xk= 2 f[xk]= 5 f'[x2]= 8

k=2 xk= 1 f[xk]= 2 f'[x2]= -1

k=3 xk= $\frac{6}{5}$ f[xk]= $\frac{241}{125}$ f'[x2]= $\frac{8}{25}$

k=4 xk= $\frac{37}{32}$ f[xk]= $\frac{62\,941}{32\,768}$ f'[x2]= $\frac{11}{1024}$

k=5 xk= $\frac{3286}{2845}$ f[xk]= $\frac{44\,231\,198\,681}{23\,027\,501\,125}$ f'[x2]= $\frac{17\,288}{8\,094\,025}$

```
FindMinimum[f[x], {x, 1}]
Solve[f'[x] == 0, x]
Solve[f'[x] == 0, x] // N
```

{1.9208, {x → 1.1547}}

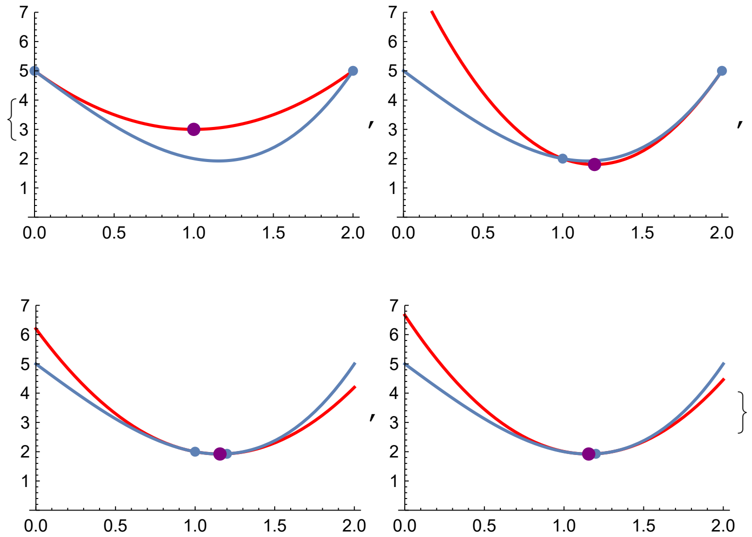
$\left\{\left\{x \to -\frac{2}{\sqrt{3}}\right\}, \left\{x \to \frac{2}{\sqrt{3}}\right\}\right\}$

{{x → -1.1547}, {x → 1.1547}}

Ilustracija :

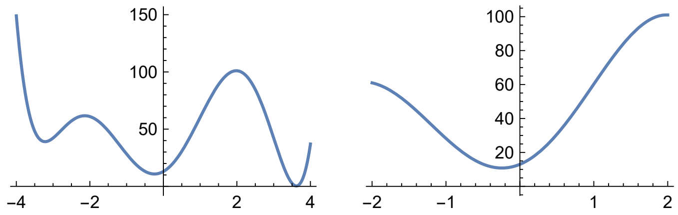
```
x0 = 0.; x1 = 2.;
eps = 0.005;
k = 2;
slpar = {};
Print["k=", 0, "    xk= ", x0, "    f[xk]= ", f[x0], "    f'[x2]= ", f'[x0]]
Print["k=", 1, "    xk= ", x1, "    f[xk]= ", f[x1], "    f'[x2]= ", f'[x1]]
While[True,
 {al, be, ga} = {alfa, beta, gama} /.
   Solve[alfa * x0^2 + beta * x0 + gama == f[x0] && alfa * x1^2 + beta * x1 + gama ==
       f[x1] && 2 * alfa * x0 + beta == f'[x0], {alfa, beta, gama}][[1]];
 x2 = x0 - 1 / 2 * ((x0 - x1) * f'[x0]) / (f'[x0] - (f[x1] - f[x0]) / (x1 - x0));
 AppendTo[slpar, Show[{Plot[al * x^2 + be * x + ga, {x, 0, 2},
      PlotStyle → Red, PlotRange → {0, 7}, AxesOrigin → {0, 0}], slf,
     ListPlot[{{x0, f[x0]}, {x1, f[x1]}}, PlotStyle → PointSize[0.03]], ListPlot[
      {{x2, al * x2^2 + be * x2 + ga}}, PlotStyle → {Purple, PointSize[0.04]}]}]];
 Print["k=", k, "    xk= ", x2, "    f[xk]= ", f[x2], "    f'[x2]= ", f'[x2]];
 If[Abs[f'[x2]] < eps, Break[]];
 x0 = x1;
 x1 = x2;
 k++;
]
slpar
```

```
k=0    xk= 0.    f[xk]= 5.    f'[x2]= -4.
k=1    xk= 2.    f[xk]= 5.    f'[x2]= 8.
k=2    xk= 1.    f[xk]= 2.    f'[x2]= -1.
k=3    xk= 1.2    f[xk]= 1.928    f'[x2]= 0.32
k=4    xk= 1.15625    f[xk]= 1.92081    f'[x2]= 0.0107422
k=5    xk= 1.15501    f[xk]= 1.9208    f'[x2]= 0.0021359
```



Primjer 2.

```
q[x_] := x^3 / 2 - 6 x + 2; u0 = 3; v0 = 4;
f[x_] := (x - u0) ^2 + (q[x] - v0) ^2
slq = Plot[f[x], {x, -4, 4}];
slf = Plot[f[x], {x, -2, 2}];
GraphicsGrid[{{slq, slf}}]
```



Metoda dvije točke - Metoda I

```
x0 = .5; x1 = 1.;
eps = 0.005;
k = 2;
Print["k=", 0, "    xk= ", x0, "    f[xk]= ", f[x0], "    f'[x2]= ", f'[x0]]
Print["k=", 1, "    xk= ", x1, "    f[xk]= ", f[x1], "    f'[x2]= ", f'[x1]]
While[True,
 x2 = x0 - (x1 - x0) * f'[x0] / (f'[x1] - f'[x0]);
 Print["k=", k, "    xk= ", x2, "    f[xk]= ", f[x2], "    f'[x2]= ", f'[x2]];
 If[Abs[f'[x2]] < eps, Break[]];
 x0 = x1;
 x1 = x2;
 k++;
]
```

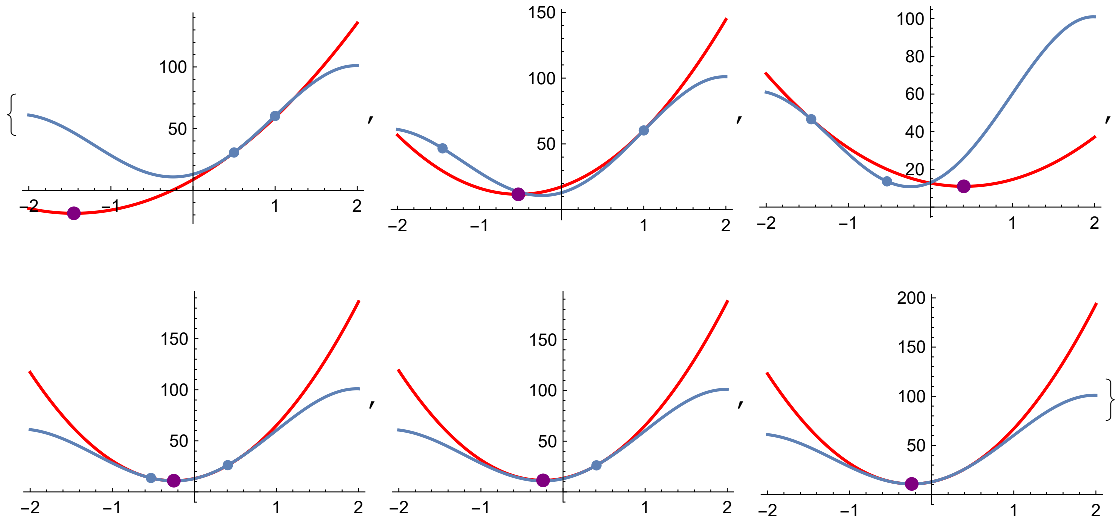
```
k=0    xk= 0.5    f[xk]= 30.6289    f'[x2]= 50.5469
k=1    xk= 1.    f[xk]= 60.25    f'[x2]= 63.5
k=2    xk= -1.45115    f[xk]= 46.6342    f'[x2]= -38.3318
k=3    xk= -0.528479    f[xk]= 13.6537    f'[x2]= -19.3027
k=4    xk= 0.40745    f[xk]= 26.1772    f'[x2]= 45.5486
k=5    xk= -0.249905    f[xk]= 10.8203    f'[x2]= -0.494542
k=6    xk= -0.242844    f[xk]= 10.8186    f'[x2]= 0.0181477
k=7    xk= -0.243094    f[xk]= 10.8186    f'[x2]= -0.0000194703
```

```
FindMinimum[f[x], {x, 1}]
{10.8186, {x → -0.243094}}
```

Ilustracija :

```
x0 = .5; x1 = 1.;
eps = 0.005;
k = 2;
slpar = {};
Print["k=", 0, "    xk= ", x0, "    f[xk]= ", f[x0], "    f'[x2]= ", f'[x0]]
Print["k=", 1, "    xk= ", x1, "    f[xk]= ", f[x1], "    f'[x2]= ", f'[x1]]
While[True,
 {al, be, ga} = {alfa, beta, gama} /.
   Solve[alfa * x0^2 + beta * x0 + gama == f[x0] && 2 * alfa * x0 + beta == f'[x0] &&
      2 * alfa * x1 + beta == f'[x1], {alfa, beta, gama}][[1]];
 x2 = x0 - (x1 - x0) * f'[x0] / (f'[x1] - f'[x0]);
 AppendTo[slpar, Show[{Plot[al * x^2 + be * x + ga, {x, -2, 2},
     PlotStyle → Red, PlotRange → All, AxesOrigin → {0, 0}], slf,
    ListPlot[{{x0, f[x0]}, {x1, f[x1]}}, PlotStyle → PointSize[0.03]], ListPlot[
     {{x2, al * x2^2 + be * x2 + ga}}, PlotStyle → {Purple, PointSize[0.04]}]}]];
 Print["k=", k, "    xk= ", x2, "    f[xk]= ", f[x2], "    f'[x2]= ", f'[x2]];
 If[Abs[f'[x2]] < eps, Break[]];
 x0 = x1;
 x1 = x2;
 k++;
]
slpar
```

```
k=0   xk= 0.5   f[xk]= 30.6289   f'[x2]= 50.5469
k=1   xk= 1.   f[xk]= 60.25   f'[x2]= 63.5
k=2   xk= -1.45115   f[xk]= 46.6342   f'[x2]= -38.3318
k=3   xk= -0.528479   f[xk]= 13.6537   f'[x2]= -19.3027
k=4   xk= 0.40745   f[xk]= 26.1772   f'[x2]= 45.5486
k=5   xk= -0.249905   f[xk]= 10.8203   f'[x2]= -0.494542
k=6   xk= -0.242844   f[xk]= 10.8186   f'[x2]= 0.0181477
k=7   xk= -0.243094   f[xk]= 10.8186   f'[x2]= -0.0000194703
```



Metoda dvije točke - Metoda II

```
x0 = .5; x1 = 1.;
eps = 0.005;
k = 2;
Print["k=", 0, "    xk= ", x0, "    f[xk]= ", f[x0], "    f'[x2]= ", f'[x0]]
Print["k=", 1, "    xk= ", x1, "    f[xk]= ", f[x1], "    f'[x2]= ", f'[x1]]
While[True,
 x2 = x0 - 1 / 2 * ((x0 - x1) * f'[x0]) / (f'[x0] - (f[x1] - f[x0]) / (x1 - x0));
 Print["k=", k, "    xk= ", x2, "    f[xk]= ", f[x2], "    f'[x2]= ", f'[x2]];
 If[Abs[f'[x2]] < eps, Break[]];
 x0 = x1;
 x1 = x2;
 k++;
]
```

```
k=0    xk= 0.5    f[xk]= 30.6289    f'[x2]= 50.5469
k=1    xk= 1.    f[xk]= 60.25    f'[x2]= 63.5
k=2    xk= -0.953279    f[xk]= 26.4297    f'[x2]= -38.3851
k=3    xk= -0.342776    f[xk]= 11.1755    f'[x2]= -7.11089
k=4    xk= -0.0787891    f[xk]= 11.8122    f'[x2]= 12.1441
k=5    xk= -0.244215    f[xk]= 10.8187    f'[x2]= -0.0814707
k=6    xk= -0.242434    f[xk]= 10.8187    f'[x2]= 0.0479535
k=7    xk= -0.243094    f[xk]= 10.8186    f'[x2]= -8.33644×10^-7
```

```
FindMinimum[f[x], {x, 1}]
```

```
{10.8186, {x → -0.243094}}
```

Ilustracija :

```
x0 = .5; x1 = 1.;
eps = 0.005;
k = 2;
slpar = {};
Print["k=", 0, "    xk= ", x0, "    f[xk]= ", f[x0], "    f'[x2]= ", f'[x0]]
Print["k=", 1, "    xk= ", x1, "    f[xk]= ", f[x1], "    f'[x2]= ", f'[x1]]
While[True,
 {al, be, ga} = {alfa, beta, gama} /.
   Solve[alfa * x0^2 + beta * x0 + gama == f[x0] && alfa * x1^2 + beta * x1 + gama ==
       f[x1] && 2 * alfa * x0 + beta == f'[x0], {alfa, beta, gama}][[1]];
 x2 = x0 - 1 / 2 * ((x0 - x1) * f'[x0]) / (f'[x0] - (f[x1] - f[x0]) / (x1 - x0));
 AppendTo[slpar, Show[{Plot[al * x^2 + be * x + ga, {x, -2, 2},
      PlotStyle → Red, PlotRange → All, AxesOrigin → {0, 0}], slf,
     ListPlot[{{x0, f[x0]}, {x1, f[x1]}}, PlotStyle → PointSize[0.03]], ListPlot[
      {{x2, al * x2^2 + be * x2 + ga}}, PlotStyle → {Purple, PointSize[0.04]}]}]];
 Print["k=", k, "    xk= ", x2, "    f[xk]= ", f[x2], "    f'[x2]= ", f'[x2]];
 If[Abs[f'[x2]] < eps, Break[]];
 x0 = x1;
 x1 = x2;
 k++;
]
slpar
```

```
k=0   xk= 0.5   f[xk]= 30.6289   f'[x2]= 50.5469

k=1   xk= 1.   f[xk]= 60.25   f'[x2]= 63.5

k=2   xk= -0.953279   f[xk]= 26.4297   f'[x2]= -38.3851

k=3   xk= -0.342776   f[xk]= 11.1755   f'[x2]= -7.11089

k=4   xk= -0.0787891   f[xk]= 11.8122   f'[x2]= 12.1441

k=5   xk= -0.244215   f[xk]= 10.8187   f'[x2]= -0.0814707

k=6   xk= -0.242434   f[xk]= 10.8187   f'[x2]= 0.0479535
```

k=7   xk= -0.243094   f[xk]= 10.8186   f'[x2]= -8.33644×10^{-7}