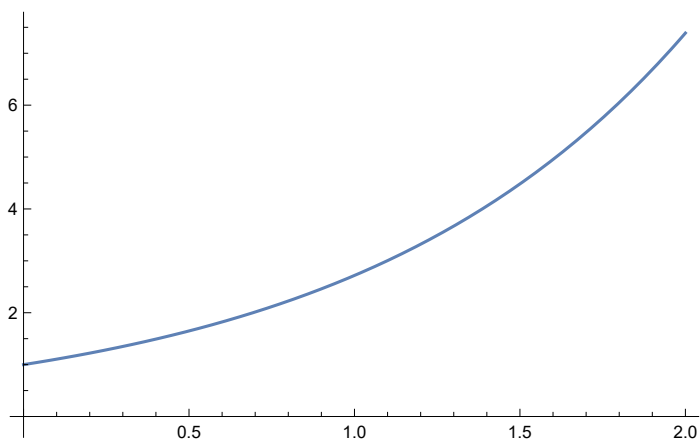
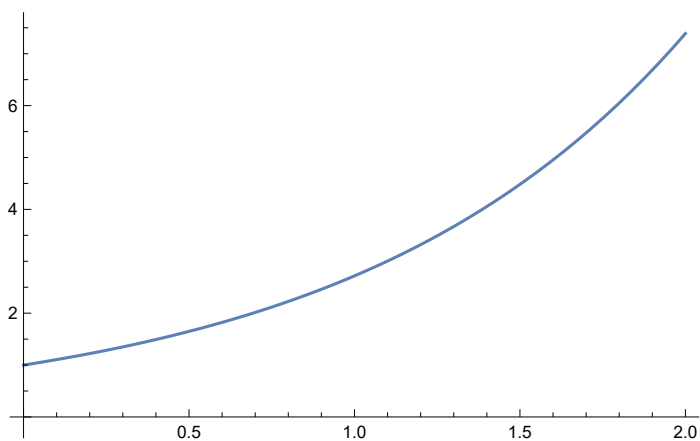


# Vježbe 6. Lipschitz neprekidne funkcije, metoda Pijavskog, Shubertova metoda

## Lipschitz neprekidne funkcije

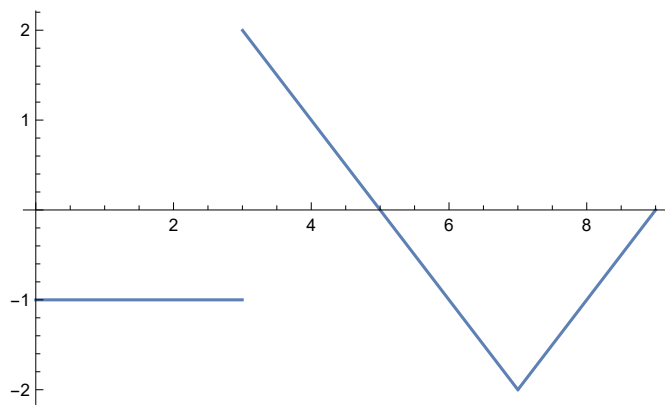
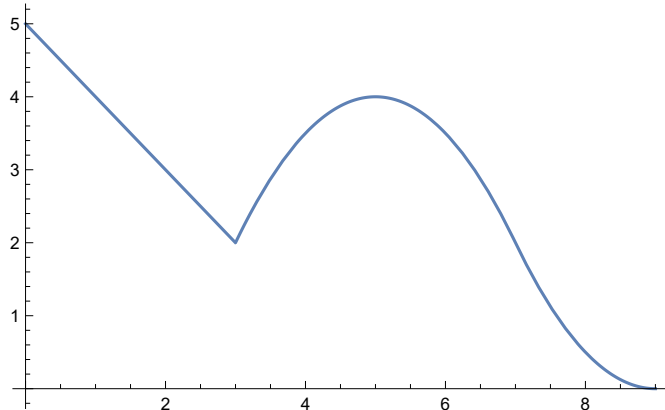
(a)

```
f[x_] := Exp[x]  
Plot[f[x], {x, 0, 2}, AxesOrigin -> {0, 0}]  
Plot[f'[x], {x, 0, 2}, AxesOrigin -> {0, 0}]
```



(b)

```
f[x_] := Piecewise[
  {{5 - x, x ≤ 3}, {4 - 1/2 * (x - 5)^2, 3 < x ≤ 7}, {1/2 * (x - 9)^2, 7 < x ≤ 9}}]
Plot[f[x], {x, 0, 9}, Exclusions → None]
Plot[f'[x], {x, 0, 9}]
```



```
Limit[(4 - 1/2 * (7 + h - 5)^2 - (4 - 1/2 * (7 - 5)^2)) / h, h → 0]
```

```
Limit[(1/2 * (7 + h - 9)^2 - (1/2 * (7 - 9)^2)) / h, h → 0]
```

-2

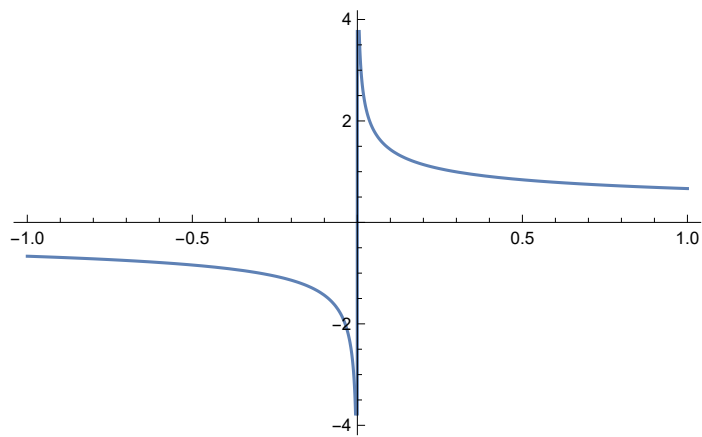
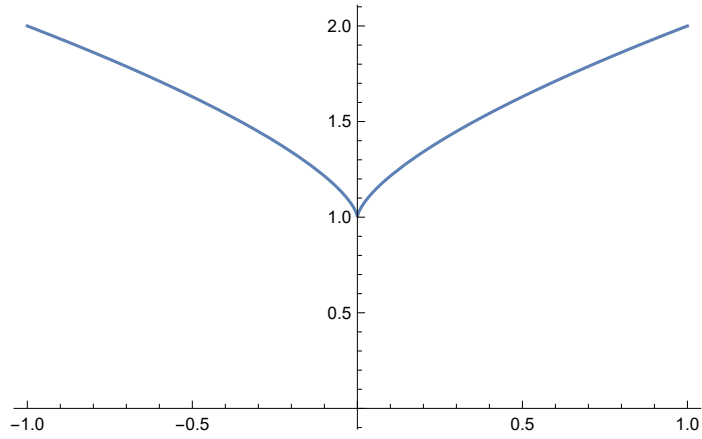
-2

(c)

```
f[x_] := 1 + (x^2)^(1/3)
```

```
Plot[f[x], {x, -1, 1}, AxesOrigin -> {0, 0}]
```

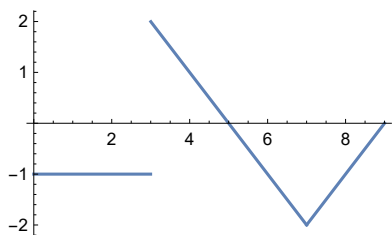
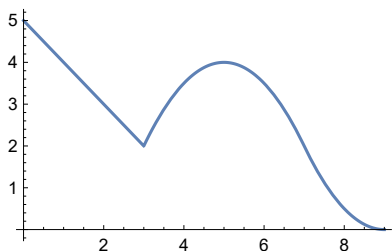
```
Plot[f'[x], {x, -1, 1}, AxesOrigin -> {0, 0}]
```



## Metoda slomljenih pravaca (Pijavski)

### Primjer I

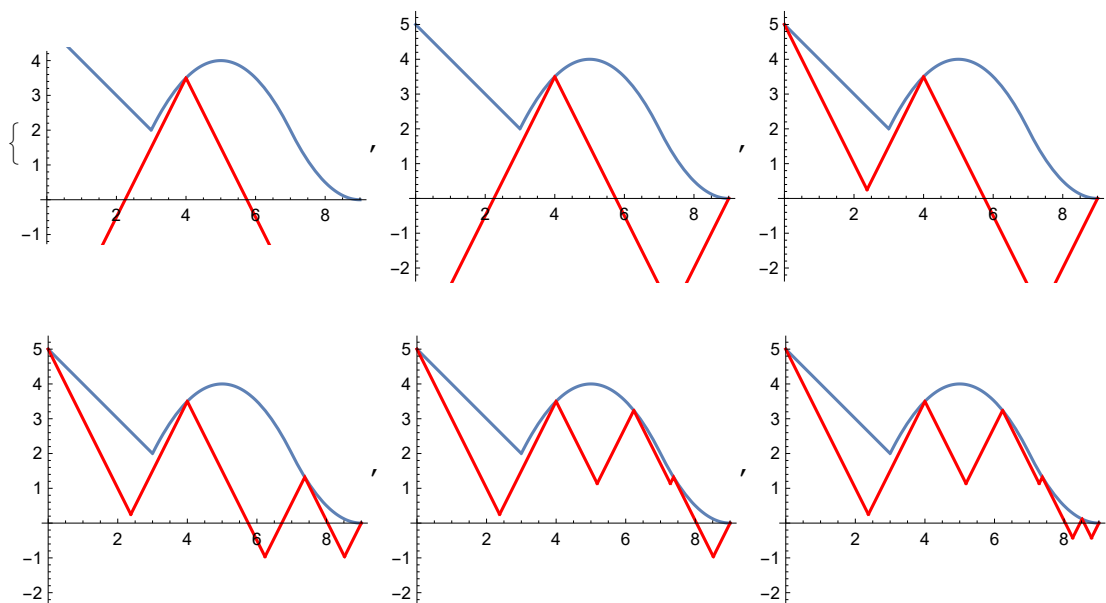
```
f[x_] := Piecewise[  
  {{5 - x, x ≤ 3}, {4 - 1/2 * (x - 5)^2, 3 < x ≤ 7}, {1/2 * (x - 9)^2, 7 < x ≤ 9}}]  
a = 0; b = 9;  
s11 = Plot[f[x], {x, 0, 9}, Exclusions → None]  
L = 2;  
u0 = 4;  
Plot[f'[x], {x, a, b}]
```



```

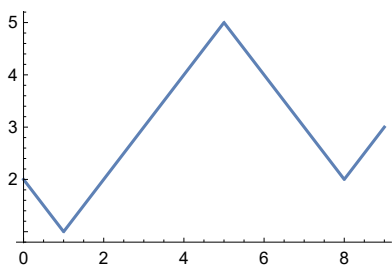
uovi = {u0}; (*u ovaj vektor spremaju se aproksimacije u[[i]]*)
briter = 5;
slike = Table[0, {i, 1, briter + 1}];
(* Iteracija 1 *)
K[u_, v_] := f[v] - L * Abs[u - v];
P[u_] := Max[Table[K[u, uovi[[i]]], {i, Length[uovi]}]];
slike[[1]] = Show[{s11, Plot[P[u], {u, a, b}, PlotStyle → Red]},
  PlotRange → {-1, 4}, PlotRange → {0, 4}, AspectRatio → Automatic];
vovi = {{a, K[a, u0]}, {b, K[b, u0]}}; (*služi određivanju minimuma funkcije P -
  bilježi sve "donje vrhove"*)
(* Iterativni postupak *)
Do[
  (*nađi minimum za P - novi u*)
  k = Ordering[vovi[[All, 2]], 1][[1]]; (*pozicija minimuma*)
  min = vovi[[k]];
  vovi = Delete[vovi, k];
  (*Taj više nije kandidat za minimum od P*)
  AppendTo[uovi, min[[1]]];
  (*nacrtaj novu funkciju P - s novom tockom u*)
  slike[[kk + 1]] = Show[{s11, Plot[P[u], {u, a, b}, PlotStyle → Red]},
    PlotRange → {-2, 5}, AspectRatio → Automatic];
  (*Dodali smo novu tocku u, onda i novu funkciju K u definiciju P*)
  (*Zbog toga smo dobili dva nova kandidata za minimum od P*)
  (*To su tocke u kojima nova funkcija K sjece susjedne uove*)
  s1 = a - 1;
  s2 = b + 1;
  Do[
    If[s1 < uovi[[i]] < uovi[[-1]], s1 = uovi[[i]];
    If[uovi[[-1]] < uovi[[i]] < s2, s2 = uovi[[i]];
    , {i, Length[uovi] - 1}];
  If[a ≤ s1 ≤ b,
    AppendTo[vovi, {x, y} /. Solve[{(y - f[s1]) == -L (x - s1),
      (y - f[uovi[[-1]]) == L (x - uovi[[-1]])}, {x, y}][[1]]];
  ];
  If[a ≤ s2 ≤ b, AppendTo[vovi, {x, y} /. Solve[{(y - f[s2]) == L (x - s2),
    (y - f[uovi[[-1]]) == -L (x - uovi[[-1]])}, {x, y}][[1]]];
  ];
  , {kk, briter}];
(* Rezultati *)
Print["Minimum: ", N[{Last[uovi], f[Last[uovi]]}]]
Print["Iteracije: ",
  N[Transpose[Table[{uovi[[i]], f[uovi[[i]]}], {i, briter + 1}]]] // TableForm]
slike
Minimum: {8.51758, 0.116365}
Iteracije: 4.    9.    0.    7.375    6.23242    8.51758
             3.5   0.    5.    1.32031  3.24057    0.116365

```



## Primjer 2

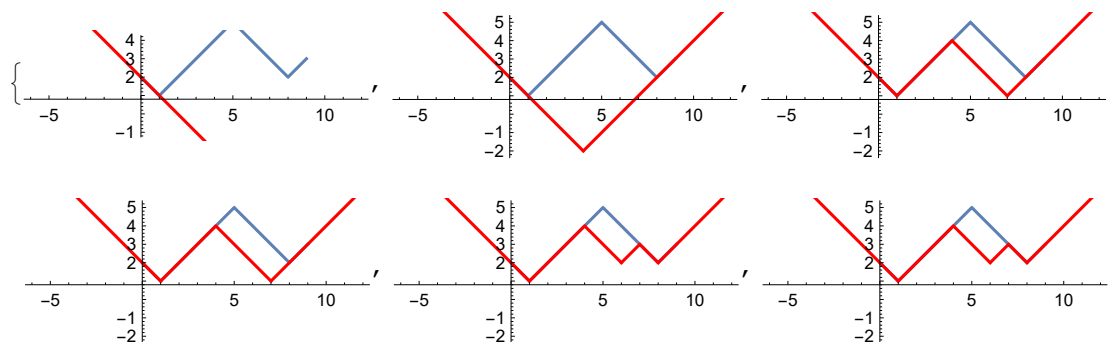
```
f[x_] := Min[{Abs[x - 1] + 1, Abs[x - 8] + 2}]
a = -6; b = 12;
s11 = Plot[f[x], {x, 0, 9}, Exclusions -> None]
L = 1;
u0 = -6;
```



```

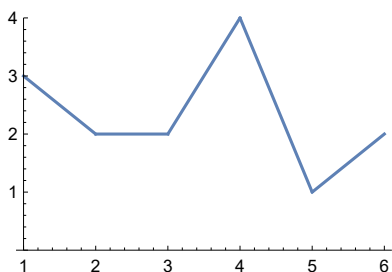
uovi = {u0}; (*u ovaj vektor spremaju se aproksimacije u[[i]]*)
briter = 5;
slike = Table[0, {i, 1, briter + 1}];
(* Iteracija 1 *)
K[u_, v_] := f[v] - L * Abs[u - v];
P[u_] := Max[Table[K[u, uovi[[i]]], {i, Length[uovi]}]];
slike[[1]] = Show[{s11, Plot[P[u], {u, a, b}, PlotStyle → Red]},
  PlotRange → {-1, 4}, PlotRange → {0, 4}, AspectRatio → Automatic];
vovi = {{a, K[a, u0]}, {b, K[b, u0]}}; (*služi određivanju minimuma funkcije P -
  bilježi sve "donje vrhove"*)
(* Iterativni postupak *)
Do[
  (*nađi minimum za P - novi u*)
  k = Ordering[vovi[[All, 2]], 1][[1]]; (*pozicija minimuma*)
  min = vovi[[k]];
  vovi = Delete[vovi, k];
  (*Taj više nije kandidat za minimum od P*)
  AppendTo[uovi, min[[1]]];
  (*nacrtaj novu funkciju P - s novom tockom u*)
  slike[[kk + 1]] = Show[{s11, Plot[P[u], {u, a, b}, PlotStyle → Red]},
    PlotRange → {-2, 5}, AspectRatio → Automatic];
  (*Dodali smo novu tocku u, onda i novu funkciju K u definiciju P*)
  (*Zbog toga smo dobili dva nova kandidata za minimum od P*)
  (*To su tocke u kojima nova funkcija K sjece susjedne uove*)
  s1 = a - 1;
  s2 = b + 1;
  Do[
    If[s1 < uovi[[i]] < uovi[[-1]], s1 = uovi[[i]];
    If[uovi[[-1]] < uovi[[i]] < s2, s2 = uovi[[i]];
    , {i, Length[uovi] - 1}];
  If[a ≤ s1 ≤ b,
    AppendTo[vovi, {x, y} /. Solve[{(y - f[s1]) == -L (x - s1),
      (y - f[uovi[[-1]]) == L (x - uovi[[-1]])}, {x, y}][[1]]];
  ];
  If[a ≤ s2 ≤ b, AppendTo[vovi, {x, y} /. Solve[{(y - f[s2]) == L (x - s2),
    (y - f[uovi[[-1]]) == -L (x - uovi[[-1]])}, {x, y}][[1]]];
  ];
  , {kk, briter}];
(* Rezultati *)
Print["Minimum: ", N[{Last[uovi], f[Last[uovi]]}]]
Print["Iteracije: ",
  N[Transpose[Table[{uovi[[i]], f[uovi[[i]]}], {i, briter + 1}]]] // TableForm]
slike
Minimum: {1., 1.}
Iteracije: -6.   12.   4.   1.   7.   1.
            8.   6.   4.   1.   3.   1.

```



### Primjer 3

```
(*f[x_]:=x^2; a=-1.;b=1.;*)
f[x_] := Piecewise[{{-x+4, 1 ≤ x ≤ 2}, {2, 2 ≤ x ≤ 3},
  {2x-4, 3 ≤ x ≤ 4}, {-3x+16, 4 ≤ x ≤ 5}, {x-4, 5 ≤ x ≤ 6}}]
f[x]
a = 1; b = 6;
s11 = Plot[f[x], {x, a, b}, PlotRange → {0, 4}, ImageSize → 200]
L = 3;
u0 = 2.5;
```

$$\left\{ \begin{array}{ll} 4-x & 1 \leq x \leq 2 \\ 2 & 2 \leq x \leq 3 \\ -4+2x & 3 \leq x \leq 4 \\ 16-3x & 4 \leq x \leq 5 \\ -4+x & 5 \leq x \leq 6 \\ 0 & \text{True} \end{array} \right.$$




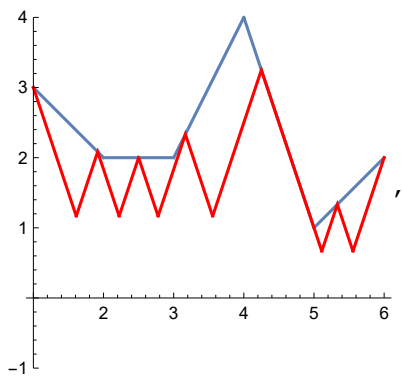
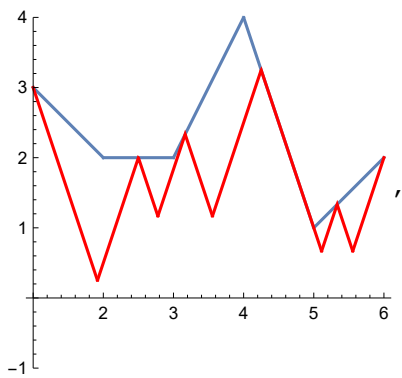
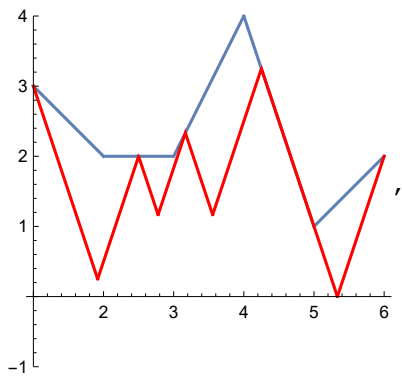
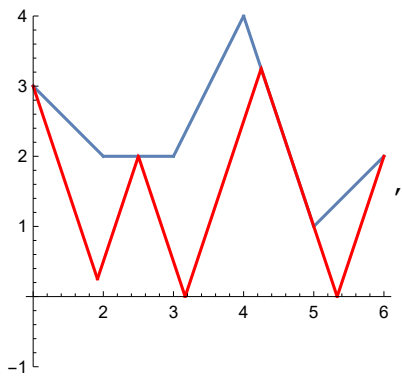
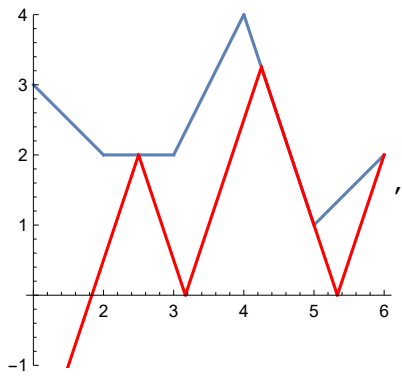
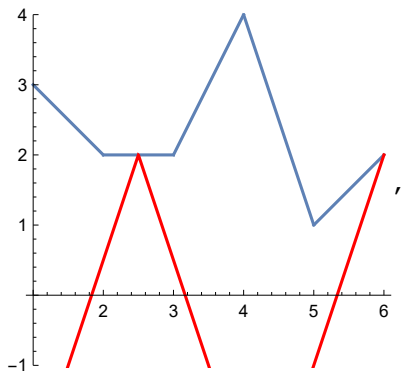
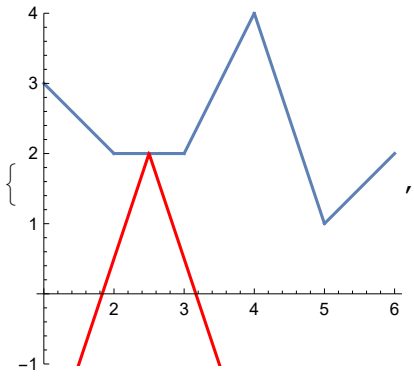
```

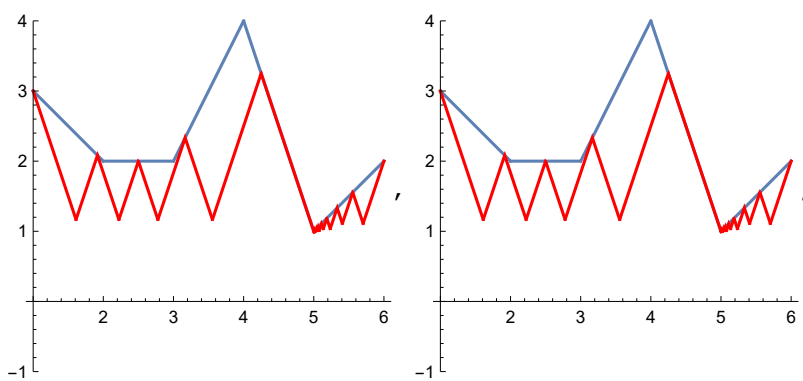
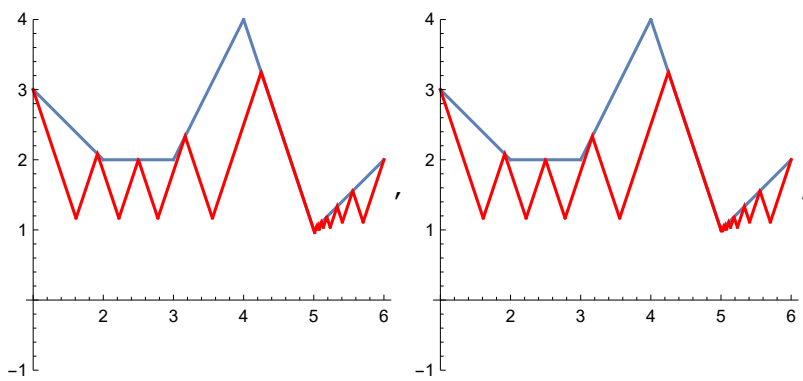
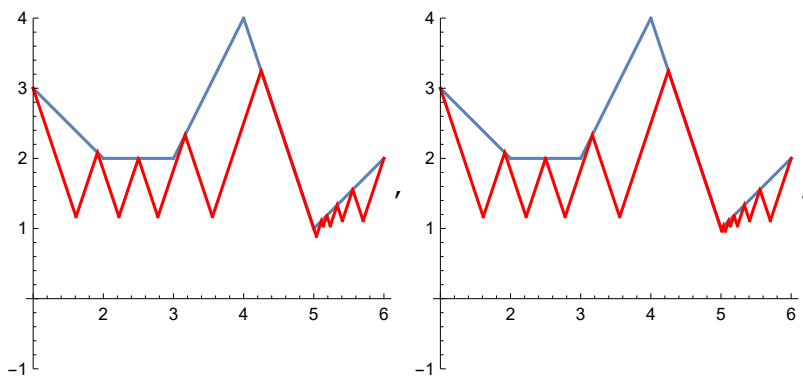
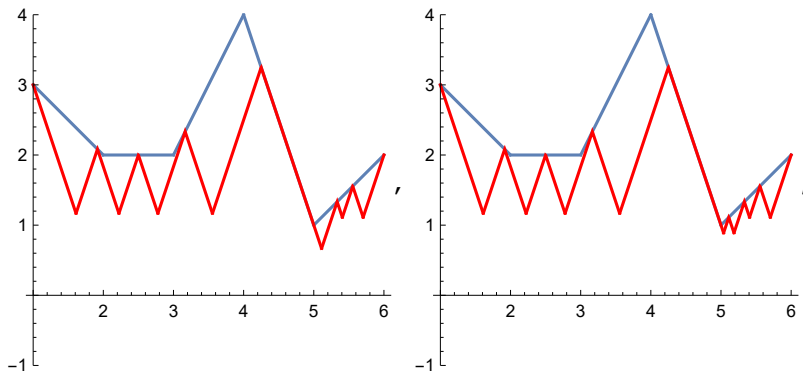
uovi = {u0}; (*u ovaj vektor spremaju se aproksimacije u[[i]]*)
briter = 30;
slike = Table[0, {i, 1, briter + 1}];
(* Iteracija 1 *)
K[u_, v_] := f[v] - L * Abs[u - v];
P[u_] := Max[Table[K[u, uovi[[i]]], {i, Length[uovi]}]];
slike[[1]] = Show[{s11, Plot[P[u], {u, a, b}, PlotStyle -> Red]},
  PlotRange -> {-1, 4}, PlotRange -> {0, 4}, AspectRatio -> Automatic];
vovi = {{a, K[a, u0]}, {b, K[b, u0]}}; (*služi određivanju minimuma funkcije P -
  bilježi sve "donje vrhove"*)
(* Iterativni postupak *)
Do[
  (*nađi minimum za P - novi u*)
  k = Ordering[vovi[[All, 2]], 1][[1]]; (*pozicija minimuma*)
  min = vovi[[k]];
  vovi = Delete[vovi, k];
  (*Taj više nije kandidat za minimum od P*)
  AppendTo[uovi, min[[1]]];
  (*nacrtaj novu funkciju P - s novom tockom u*)
  slike[[kk + 1]] = Show[{s11, Plot[P[u], {u, a, b}, PlotStyle -> Red]},
    PlotRange -> {-1, 4}, AspectRatio -> Automatic];
  (*Dodali smo novu tocku u, onda i novu funkciju K u definiciju P*)
  (*Zbog toga smo dobili dva nova kandidata za minimum od P*)
  (*To su tocke u kojima nova funkcija K sjece susjedne uove*)
  s1 = a - 1;
  s2 = b + 1;
  Do[
    If[s1 < uovi[[i]] < uovi[[-1]], s1 = uovi[[i]];
    If[uovi[[-1]] < uovi[[i]] < s2, s2 = uovi[[i]];
    , {i, Length[uovi] - 1}];
  If[a ≤ s1 ≤ b,
    AppendTo[vovi, {x, y} /. Solve[{(y - f[s1]) == -L (x - s1),
      (y - f[uovi[[-1]]) == L (x - uovi[[-1]])}, {x, y}][[1]]];
  ];
  If[a ≤ s2 ≤ b, AppendTo[vovi, {x, y} /. Solve[{(y - f[s2]) == L (x - s2),
    (y - f[uovi[[-1]]) == -L (x - uovi[[-1]])}, {x, y}][[1]]];
  ];
  , {kk, briter}];
(* Rezultati *)
Print["Minimum: ", N[{Last[uovi], f[Last[uovi]]}]]
Print["Iteracije: ",
  N[Transpose[Table[{uovi[[i]], f[uovi[[i]]}], {i, briter + 1}]]] // TableForm]
slike

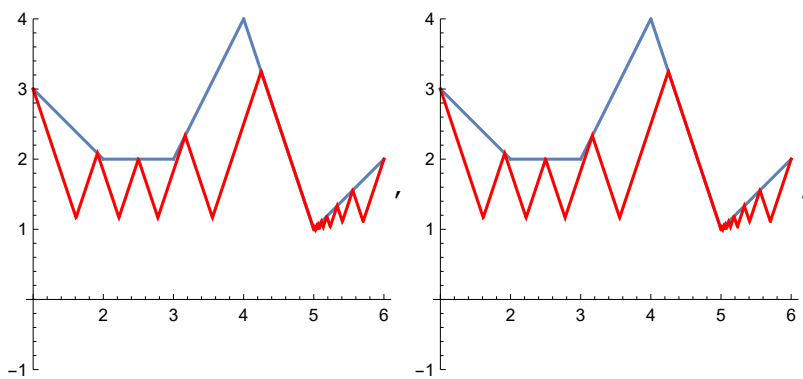
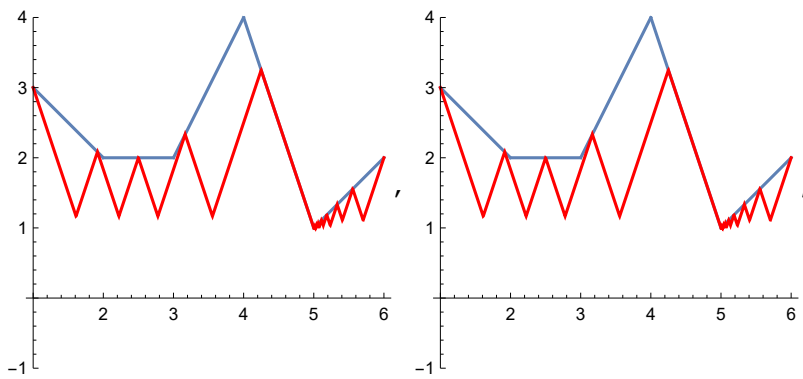
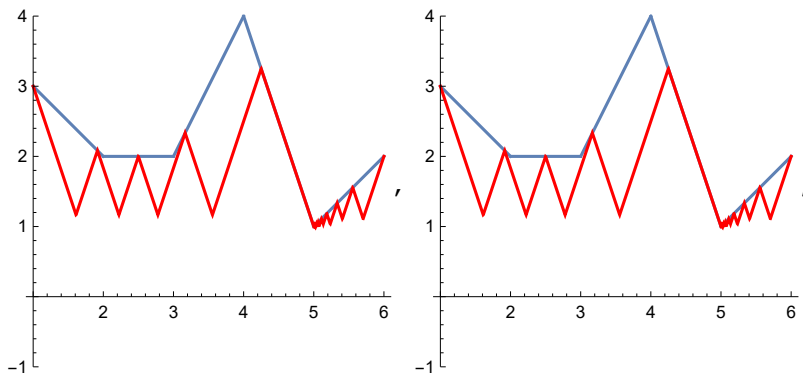
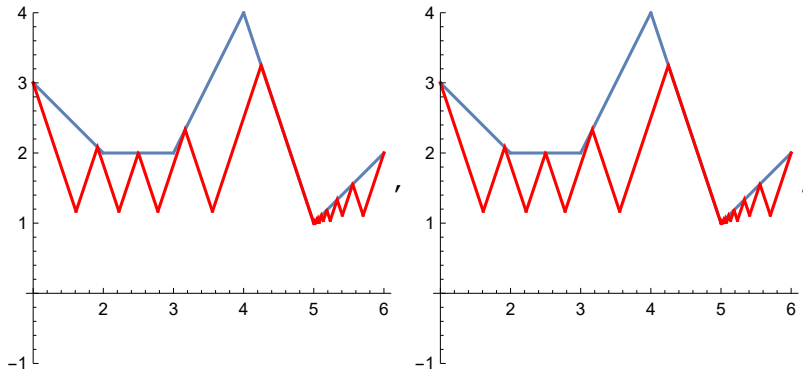
Minimum: {5., 1.}

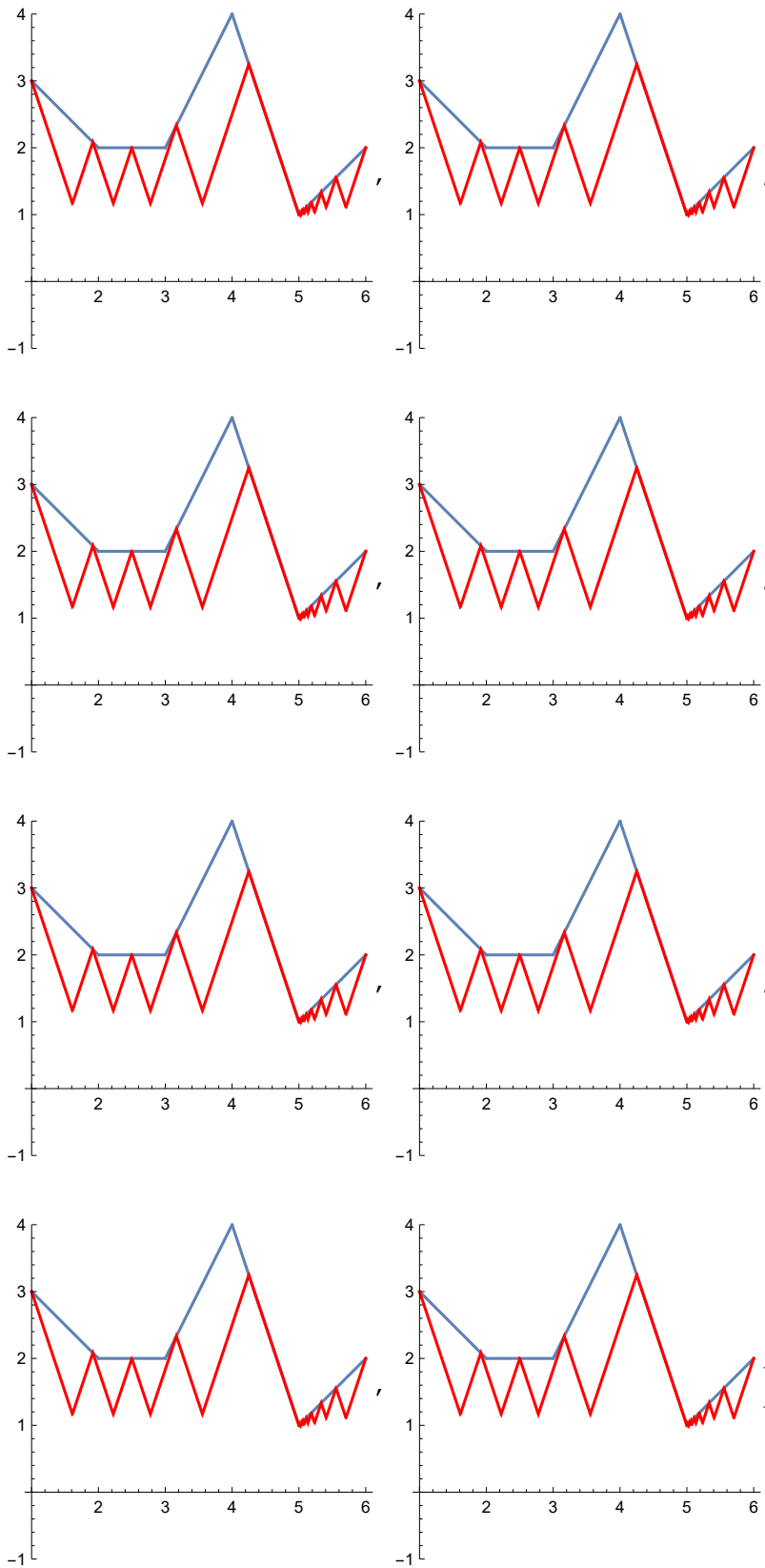
Iteracije:
  2.5   6.   4.25   1.   3.16667   5.33333   1.91667   5.55556   5.11111   5.18519
  2.    2.    3.25   3.    2.33333   1.33333   2.08333   1.55556   1.11111   1.18519

```





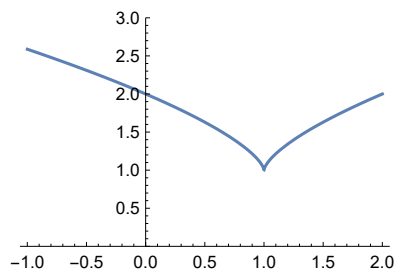




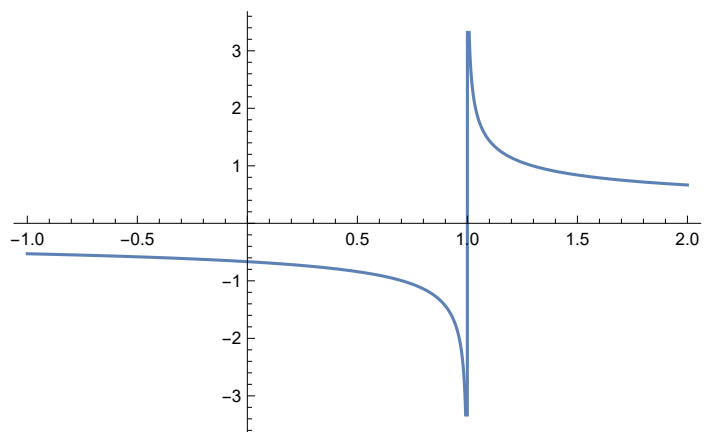
### Primjer 4

Funkcija koja nije Lipschitzova

```
a = -1; b = 2;  
f[x_] := 1 + (x - 1)^2^(1/3)  
s11 = Plot[f[x], {x, a, b}, PlotRange -> {0, 3}, ImageSize -> 200]  
a1 = 1;  
u0 = a1 * a + (1 - a1) * b;  
L = 1;  
uovi = {u0};
```



```
Plot[f'[x], {x, a, b}]
```



```

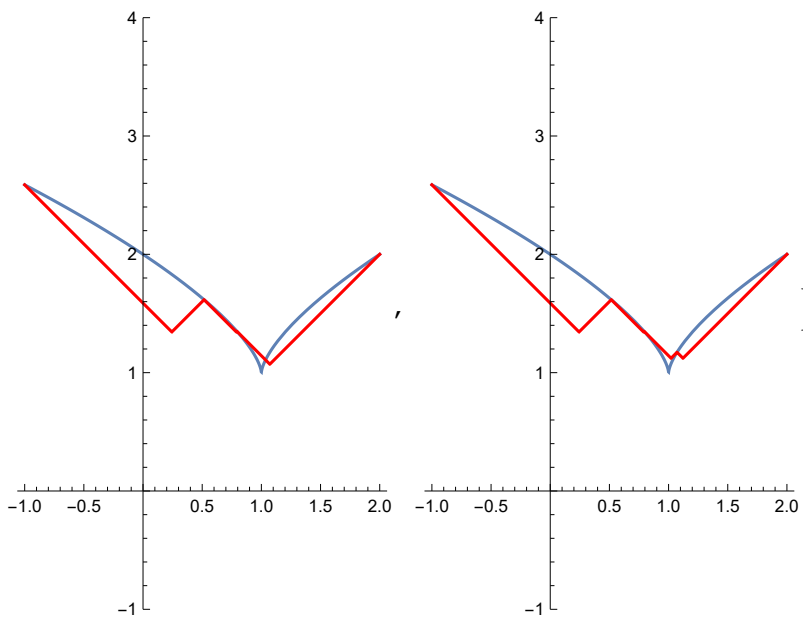
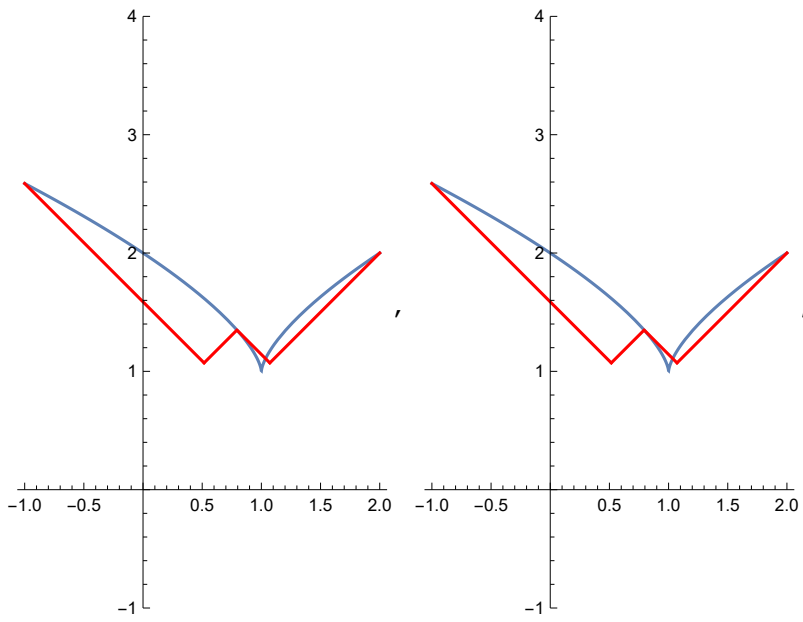
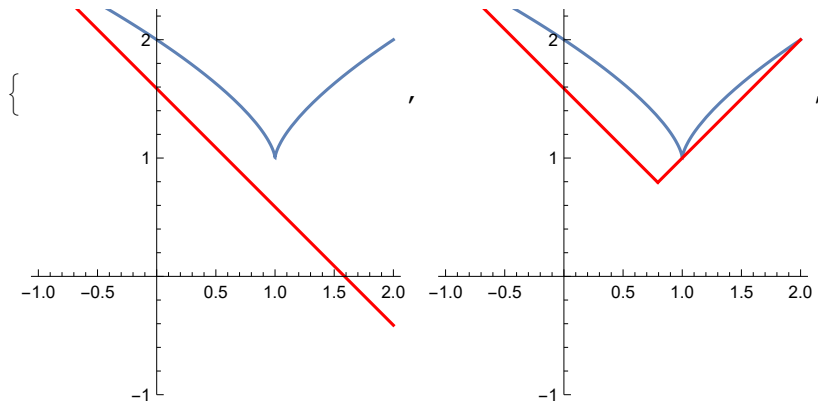
uovi = {u0}; (*u ovaj vektor spremaju se aproksimacije u[[i]]*)
briter = 5;
slike = Table[0, {i, 1, briter + 1}];
(* Iteracija 1 *)
K[u_, v_] := f[v] - L * Abs[u - v];
P[u_] := Max[Table[K[u, uovi[[i]]], {i, Length[uovi]}]];
slike[[1]] = Show[{s11, Plot[P[u], {u, a, b}, PlotStyle -> Red]},
  PlotRange -> {-1, 4}, PlotRange -> {0, 4}, AspectRatio -> Automatic];
vovi = {{a, K[a, u0]}, {b, K[b, u0]}}; (*služi odredivanju minimuma funkcije P -
  bilježi sve "donje vrhove"*)
(* Iterativni postupak *)
Do[
  (*nađi minimum za P - novi u*)
  k = Ordering[vovi[[All, 2]], 1][[1]]; (*pozicija minimuma*)
  min = vovi[[k]];
  vovi = Delete[vovi, k];
  (*Taj više nije kandidat za minimum od P*)
  AppendTo[uovi, min[[1]]];
  (*nacrtaj novu funkciju P - s novom tockom u*)
  slike[[kk + 1]] = Show[{s11, Plot[P[u], {u, a, b}, PlotStyle -> Red]},
    PlotRange -> {-1, 4}, AspectRatio -> Automatic];
  (*Dodali smo novu tocku u, onda i novu funkciju K u definiciju P*)
  (*Zbog toga smo dobili dva nova kandidata za minimum od P*)
  (*To su tocke u kojima nova funkcija K sjece susjedne uove*)
  s1 = a - 1;
  s2 = b + 1;
  Do[
    If[s1 < uovi[[i]] < uovi[[-1]], s1 = uovi[[i]];
    If[uovi[[-1]] < uovi[[i]] < s2, s2 = uovi[[i]];
    , {i, Length[uovi] - 1}];
  If[a ≤ s1 ≤ b,
    AppendTo[vovi, {x, y} /. Solve[{(y - f[s1]) == -L (x - s1),
      (y - f[uovi[[-1]]) == L (x - uovi[[-1]])}, {x, y}][[1]]];
  ];
  If[a ≤ s2 ≤ b, AppendTo[vovi, {x, y} /. Solve[{(y - f[s2]) == L (x - s2),
    (y - f[uovi[[-1]]) == -L (x - uovi[[-1]])}, {x, y}][[1]]];
  ];
  , {kk, briter}];
(* Rezultati *)
Print["Minimum: ", N[{Last[uovi], f[Last[uovi]]}]]
Print["Iteracije: ",
  N[Transpose[Table[{uovi[[i]], f[uovi[[i]]}], {i, briter + 1}]] // TableForm]
slike

```

Minimum: {1.07142, 1.17214}

Iteracije:	-1.	2.	0.793701	-1.	0.515981	1.07142
	2.5874	2.	1.34914	2.5874	1.61646	1.17214



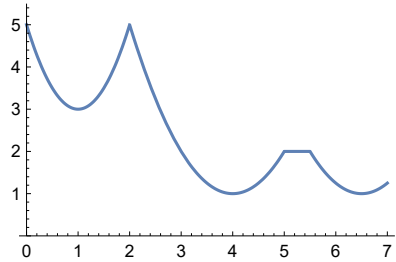


### Primjer 5

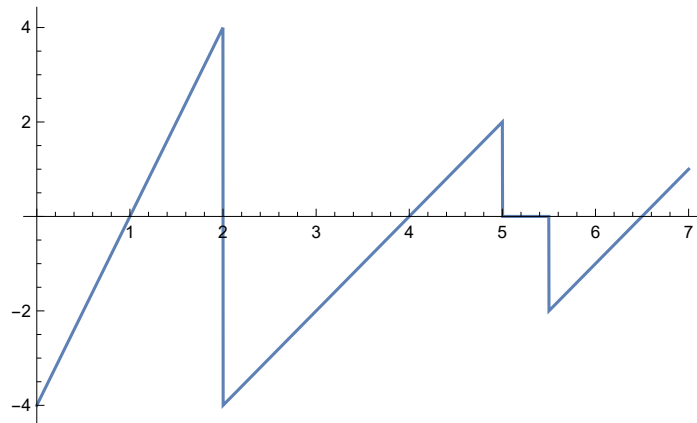


Argmin je dvoclani skup

```
f[x_] :=
  If[x ≤ 2, 2 (x - 1)^2 + 3, If[x ≤ 5, (x - 4)^2 + 1, If[x ≤ 5.5, 2, (x - 6.5)^2 + 1]]]
a = 0; b = 7; yb = 0; yt = 5.5;
al = 1;
u0 = al * a + (1 - al) * b;
s11 = Plot[f[x], {x, a, b}, PlotRange → {yb, yt}, ImageSize → 200]
L = 4;
uovi = {u0};
```



```
Plot[f'[x], {x, a, b}]
```



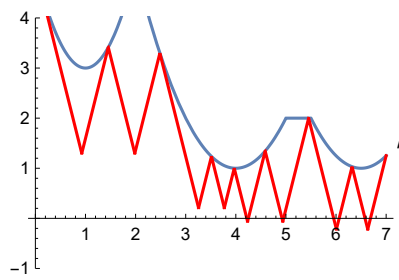
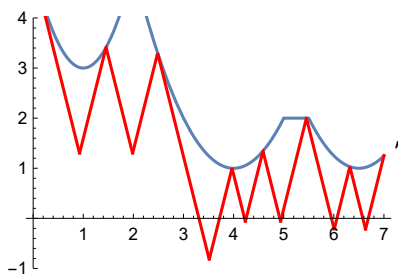
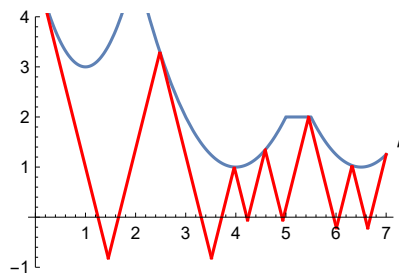
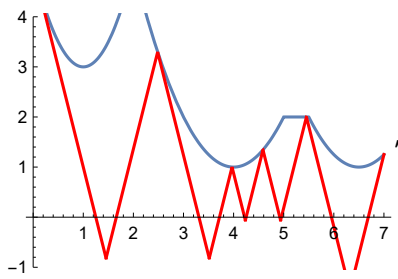
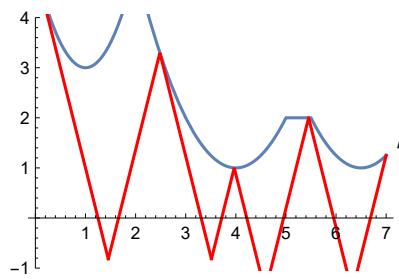
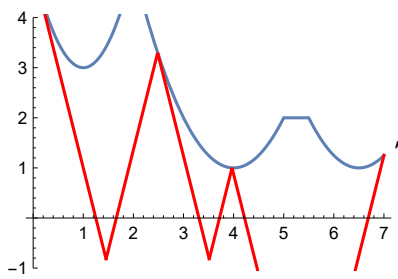
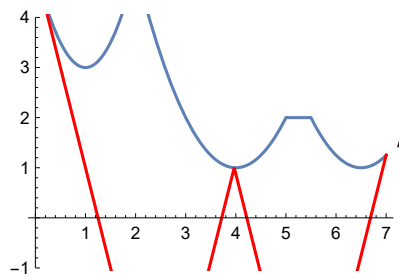
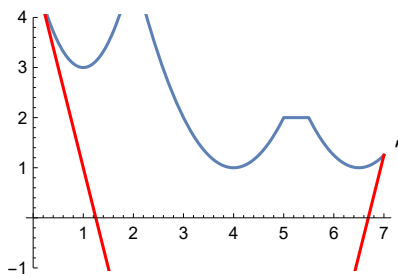
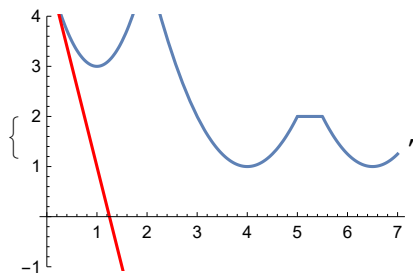
```

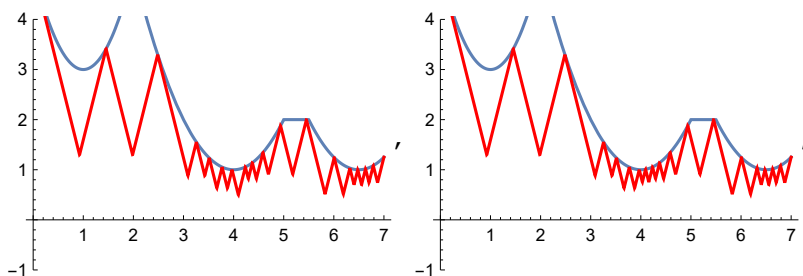
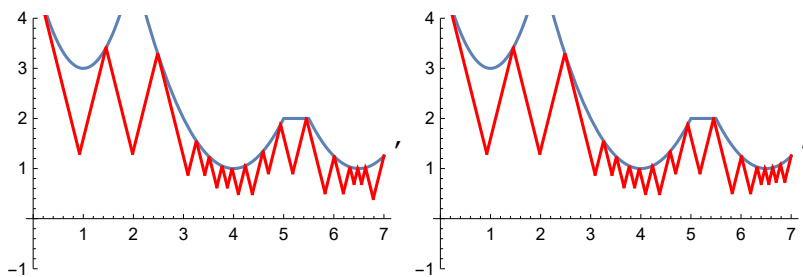
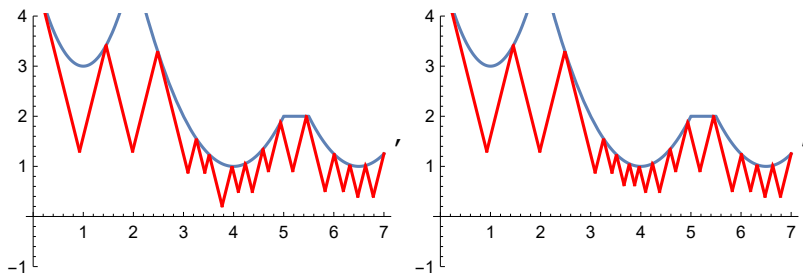
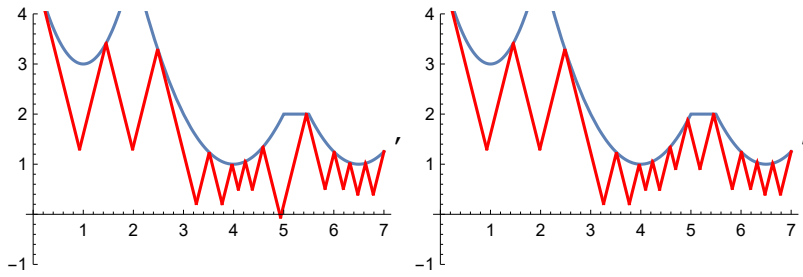
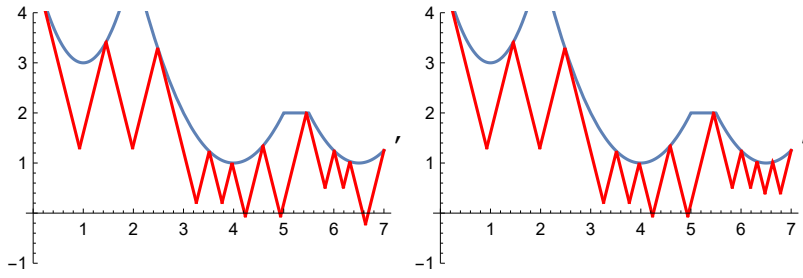
uovi = {u0}; (*u ovaj vektor spremaju se aproksimacije u[[i]]*)
briter = 20;
slike = Table[0, {i, 1, briter + 1}];
(* Iteracija 1 *)
K[u_, v_] := f[v] - L * Abs[u - v];
P[u_] := Max[Table[K[u, uovi[[i]]], {i, Length[uovi]}]];
slike[[1]] = Show[{s11, Plot[P[u], {u, a, b}, PlotStyle → Red]},
  PlotRange → {-1, 4}, PlotRange → {0, 4}, AspectRatio → Automatic];
vovi = {{a, K[a, u0]}, {b, K[b, u0]}}; (*služi određivanju minimuma funkcije P -
  bilježi sve "donje vrhove"*)
(* Iterativni postupak *)
Do[
  (*nađi minimum za P - novi u*)
  k = Ordering[vovi[[All, 2]], 1][[1]]; (*pozicija minimuma*)
  min = vovi[[k]];
  vovi = Delete[vovi, k];
  (*Taj više nije kandidat za minimum od P*)
  AppendTo[uovi, min[[1]]];
  (*nacrtaj novu funkciju P - s novom tockom u*)
  slike[[kk + 1]] = Show[{s11, Plot[P[u], {u, a, b}, PlotStyle → Red]},
    PlotRange → {-1, 4}, AspectRatio → Automatic];
  (*Dodali smo novu tocku u, onda i novu funkciju K u definiciju P*)
  (*Zbog toga smo dobili dva nova kandidata za minimum od P*)
  (*To su tocke u kojima nova funkcija K sjece susjedne uove*)
  s1 = a - 1;
  s2 = b + 1;
  Do[
    If[s1 < uovi[[i]] < uovi[[-1]], s1 = uovi[[i]];
    If[uovi[[-1]] < uovi[[i]] < s2, s2 = uovi[[i]];
    , {i, Length[uovi] - 1}];
  If[a ≤ s1 ≤ b,
    AppendTo[vovi, {x, y} /. Solve[{(y - f[s1]) == -L (x - s1),
      (y - f[uovi[[-1]]) == L (x - uovi[[-1]])}, {x, y}][[1]]];
  ];
  If[a ≤ s2 ≤ b, AppendTo[vovi, {x, y} /. Solve[{(y - f[s2]) == L (x - s2),
    (y - f[uovi[[-1]]) == -L (x - uovi[[-1]])}, {x, y}][[1]]];
  ];
  , {kk, briter}];
(* Rezultati *)
Print["Minimum: ", N[{Last[uovi], f[Last[uovi]]}]]
Print["Iteracije: ",
  N[Transpose[Table[{uovi[[i]], f[uovi[[i]]}], {i, briter + 1}]]] // TableForm]
slike

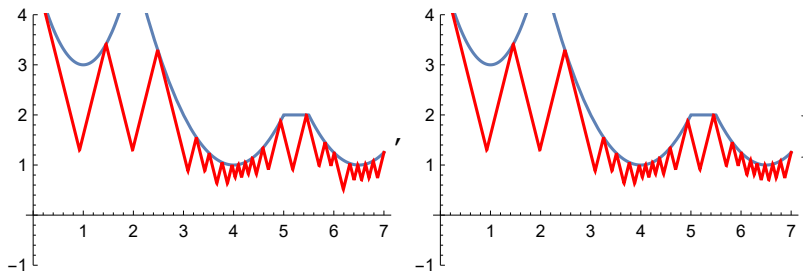
Minimum: {6.19033, 1.0959}

Iteracije:
0.    7.    3.96875  2.48425  5.45325  4.58612  6.32037  1.45494  3.51357
5.    1.25  1.00098  3.29749  2.        1.34354  1.03227  3.41394  1.23662

```





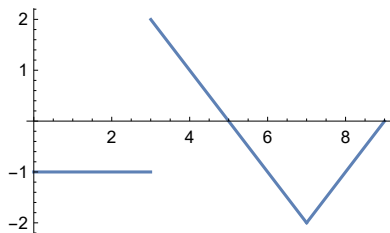
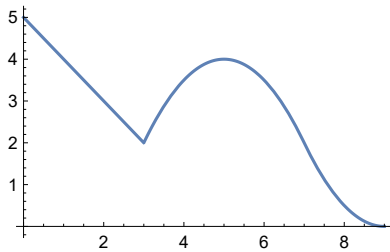


## Shubertova metoda

Shubertova metoda je ekvivalentna (do na iteraciju) s prethodnom za  $u_0 = a$

### Primjer I

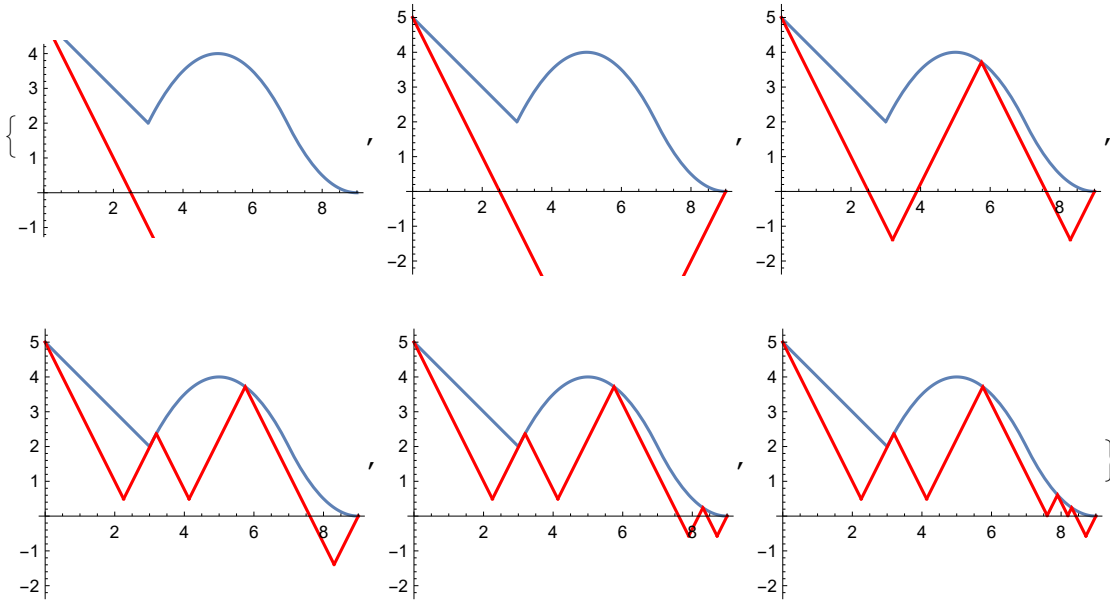
```
f[x_] := Piecewise[
  {{5 - x, x ≤ 3}, {4 - 1/2 * (x - 5)^2, 3 < x ≤ 7}, {1/2 * (x - 9)^2, 7 < x ≤ 9}}]
a = 0; b = 9;
s11 = Plot[f[x], {x, 0, 9}, Exclusions → None]
L = 2;
u0 = 0;
Plot[f'[x], {x, a, b}]
```



```

uovi = {u0}; (*u ovaj vektor spremaju se aproksimacije u[[i]]*)
briter = 5;
slike = Table[0, {i, 1, briter + 1}];
(* Iteracija 1 *)
K[u_, v_] := f[v] - L * Abs[u - v];
P[u_] := Max[Table[K[u, uovi[[i]]], {i, Length[uovi]}]];
slike[[1]] = Show[{s11, Plot[P[u], {u, a, b}, PlotStyle → Red]},
  PlotRange → {-1, 4}, PlotRange → {0, 4}, AspectRatio → Automatic];
vovi = {{a, K[a, u0]}, {b, K[b, u0]}}; (*služi određivanju minimuma funkcije P -
  bilježi sve "donje vrhove"*)
(* Iterativni postupak *)
Do[
  (*nađi minimum za P - novi u*)
  k = Ordering[vovi[[All, 2]], 1][[1]]; (*pozicija minimuma*)
  min = vovi[[k]];
  vovi = Delete[vovi, k];
  (*Taj više nije kandidat za minimum od P*)
  AppendTo[uovi, min[[1]]];
  (*nacrtaj novu funkciju P - s novom tockom u*)
  slike[[kk + 1]] = Show[{s11, Plot[P[u], {u, a, b}, PlotStyle → Red]},
    PlotRange → {-2, 5}, AspectRatio → Automatic];
  (*Dodali smo novu tocku u, onda i novu funkciju K u definiciju P*)
  (*Zbog toga smo dobili dva nova kandidata za minimum od P*)
  (*To su tocke u kojima nova funkcija K sjece susjedne uove*)
  s1 = a - 1;
  s2 = b + 1;
  Do[
    If[s1 < uovi[[i]] < uovi[[-1]], s1 = uovi[[i]];
    If[uovi[[-1]] < uovi[[i]] < s2, s2 = uovi[[i]];
    , {i, Length[uovi] - 1}];
  If[a ≤ s1 ≤ b,
    AppendTo[vovi, {x, y} /. Solve[{{(y - f[s1]) == -L (x - s1),
      (y - f[uovi[[-1]]) == L (x - uovi[[-1]])}, {x, y}][[1]]]];
  ];
  If[a ≤ s2 ≤ b, AppendTo[vovi, {x, y} /. Solve[{{(y - f[s2]) == L (x - s2),
    (y - f[uovi[[-1]]) == -L (x - uovi[[-1]])}, {x, y}][[1]]]];
  ];
  , {kk, briter}];
(* Rezultati *)
Print["Minimum: ", N[{Last[uovi], f[Last[uovi]]}]]
Print["Iteracije: ",
  N[Transpose[Table[{uovi[[i]], f[uovi[[i]]}], {i, briter + 1}]]] // TableForm]
slike
Minimum: {7.8966, 0.608747}
Iteracije: 0.    9.    5.75    3.19531    8.30469    7.8966
             5.    0.    3.71875    2.37155    0.24173    0.608747

```



$$U[a_, b_, f_, L_] := 1/2 * (a+b) + 1/(2L) * (f[a] - f[b])$$

$$B[a_, b_, f_, L_] := 1/2 * (f[a] + f[b]) - L/2 * (b - a)$$

$$U[0, 23/4, f, 2]$$

$$\frac{409}{128}$$

$$B[0, 3.1953, f, 2]$$

$$B[3.1953, 23/4, f, 2]$$

$$B[23/4, 9, f, 2]$$

$$0.490464$$

$$0.490439$$

$$-\frac{89}{64}$$

$$U[23/4, 9, f, 2]$$

$$\frac{1063}{128}$$

$$N\left[\frac{1063}{128}\right]$$

$$8.30469$$

$$B[23/4, 1063/128, f, 2]$$

$$B[1063/128, 9, f, 2]$$

$$-\frac{37647}{65536}$$

$$-\frac{37647}{65536}$$

$$\mathbf{N} \left[ -\frac{37\,647}{65\,536} \right]$$

-0.574448