SLUČAJNI PROCESI II - ISPITNI ROK 18.06.2021.

1.

$$Y_t = (B_t - t)^2 - t(t+1-2B_t) = B_t^2 - \cancel{2B_t t} + \cancel{t^2} - \cancel{t^2} - t + \cancel{2B_t t} = B_t^2 - t$$

a) Provjerimo zahtjeve iz definicije martingala:

i) $$E|B_t^2 - t| \leq EB_t^2 + t = t + t = 2t < \infty \quad \forall t \geq 0$$

ii) Radi se o kvadratnoj transformaciji Brownovog gibanja pa je to svakako adaptiran proces s obzirom na prirodnu filtraciju BG.

iii) $$E[B_t^2 - t \mid \mathcal{F}_s] = E[(B_t - B_s + B_s)^2 - t \mid \mathcal{F}_s] = E[(B_t - B_s)^2 \mid \mathcal{F}_s] - 2E[(B_t - B_s)B_s \mid \mathcal{F}_s] + E[B_s^2 \mid \mathcal{F}_s] - t$$

$$\overset{\text{nez. prirast}}{=} \underbrace{E[(B_t - B_s)^2]}_{t-s} - 2B_s \underbrace{E[B_t - B_s]}_{0} + B_s^2 - t$$

$$= (t-s) - 2B_s \cdot 0 + B_s^2 - t = B_s^2 - s$$

$\Rightarrow (Y_t, t \geq 0)$ je martingal!

b) $$Cov(Y_t, Y_s) = E[Y_t \cdot Y_s] - E[Y_t] \cdot E[Y_s]$$

$$= E[(B_t^2 - t)(B_s^2 - s)] - \underbrace{E[B_t^2 - t]}_{0} \underbrace{E[B_s^2 - s]}_{0}$$

$$= E[B_t^2 B_s^2 - B_t^2 s - t B_s^2 + ts] = E((B_t - B_s)B_s + B_s^2)^2$$

$$= E[(B_t - B_s)^2 B_s^2] + 2E[(B_t - B_s)B_s^3] + E[B_s^4]$$

$$= E(B_t - B_s)^2 \cdot EB_s^2 + 2E(B_t - B_s) \cdot EB_s^3 + EB_s^4$$

$$= (t-s) \cdot s + 0 + 3s^2$$

$$= ts + 2s^2 = (t + 2s) \cdot s$$

$$\boxed{\begin{aligned} &EB_1^4 = 3 \\ &\Rightarrow EB_s^4 = E[(s^{1/2} B_1)^4] \\ &= s^2 EB_1^4 = 3s^2 \end{aligned}}$$

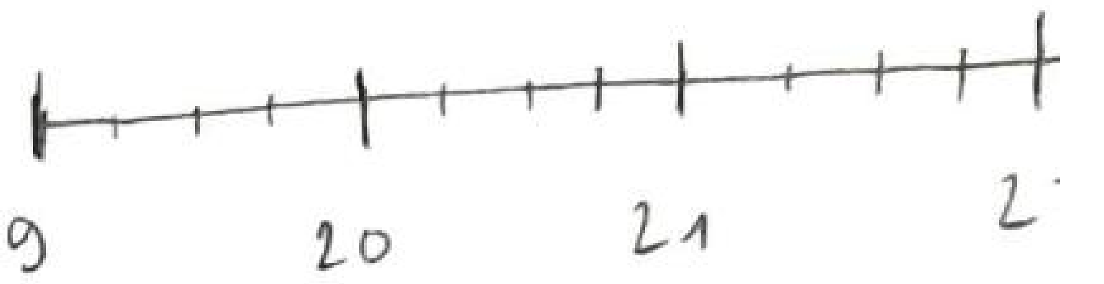
c) Ne radi se o strogo stac. procesu jer funkcija autokovarijance ne ovisi samo o $t-s$!

(2.) $(N_t^{(1)}, t \geq 0)$ - Poissonov proces koji modelira broj asova igrača A u satu s param. $\lambda_1 =$

$(N_t^{(2)}, t \geq 0)$ - ——— || ——— B u satu s param $\lambda_2 = 6$

$N_t^{(1)}$ i $N_t^{(2)}$ nezavisni!

a)



P(„u periodu od 20:15h do 21:45h tenisač A ostvaruje 5, a tenisač B 6 asova") $= P\left(N^{(1)}_{\frac{11}{4}} - N^{(1)}_{\frac{5}{4}} = 5,\ N^{(2)}_{\frac{11}{4}} - N^{(2)}_{\frac{5}{4}} = 6\right)$

$$\overset{\text{nez.}}{=} P\left(N^{(1)}_{\frac{11}{4}} - N^{(1)}_{\frac{5}{4}} = 5\right) \cdot P\left(N^{(2)}_{\frac{11}{4}} - N^{(2)}_{\frac{5}{4}} = 6\right)$$

$$= P\left(\mathcal{P}\left(\tfrac{6}{4} \cdot 5\right) = 5\right) \cdot P\left(\mathcal{P}\left(\tfrac{6}{4} \cdot 6\right) = 6\right)$$

$$= e^{-7.5} \frac{(7.5)^5}{5!} \cdot e^{-9} \frac{9^6}{6!} \approx 0.0099629667$$

b) Superpozicija Poissonovih procesa

$\Rightarrow (N_t, t \geq 0) = (N_t^{(1)} + N_t^{(2)}, t \geq 0)$ je PP s param. $\lambda_1 + \lambda_2 = 11$ i modelira broj asova u meču po satu!

$S_3 \sim \Gamma(3, 11)$

$$\Rightarrow P(S_3 \geq 1) = \int_1^\infty \frac{11^3}{\Gamma(3)} x^{3-1} e^{-11x}\,dx = \frac{1331}{2} \int_1^\infty x^2 e^{-11x}\,dx$$

$$= \frac{1331}{2}\left[\frac{x^2 e^{-11x}}{-11}\Big|_1^\infty + \frac{1}{11}\int 2x e^{-11x}\,dx\right] = \frac{1331}{2}\left[\frac{e^{-11}}{11} + \frac{2}{11}\left[\frac{e^{-11}}{11} + \frac{1}{11}\int_1 e^{-11x} dx\right.\right.$$

$$= \frac{1331}{2}\left[\frac{e^{-11}}{11} + \frac{2e^{-11}}{11^2} + \frac{e^{-11}}{11}\right] = \frac{\cancel{1331}}{2} \cdot \frac{145}{\cancel{1331}} e^{-11} = \frac{145}{2} e^{-11} \approx 0.001211$$

c) $E[W_{12}] = E[\mathcal{E}(5)] = \frac{1}{5} = 0.2$

d) $P\left(S_5^{(1)} < S_6^{(2)}\right) = \sum_{k=5}^{10} \binom{10}{k} \left(\frac{5}{11}\right)^k \left(\frac{6}{11}\right)^{10-k}$

e) $P\left(N^{(1)}_{\frac{3}{2}} = 1 \mid N_{\frac{3}{2}} = 1\right) = \frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{5}{11}$ //

(3.) • X - ML u neprekidnom vremenu, $S=\{1,2,3\}$

• $\Pi = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ $\quad q_{12}=2,\ q_{23}=1,\ q_{32}=3/2$

a) $Q = ?$, $q_{ij} = \begin{cases} -q(i), & i=j \\ q_i \pi_{ij}, & i \neq j \end{cases}$ $\Rightarrow$ $q_{13}=0$ pa je $q_{11} = -q_{12}-q_{13} = -2$

$q_{21}=0$ pa je $q_{22} = -q_{21}-q_{23} = -1$

$q_{32} = 3/2 \Rightarrow q_3 \cdot \pi_{32} = 3/2$

$\Rightarrow q_3 = 3$

$\Rightarrow q_{31} = 3 \cdot 1/2 = 3/2$

$\Rightarrow Q = \begin{bmatrix} -2 & 2 & 0 \\ 0 & -1 & 1 \\ 3/2 & 3/2 & -3 \end{bmatrix}$

b) Zanima nas vjer. $P(J_1 < 10 < J_2 \mid X_{J_1}=3, X_0=2)$

Kako je $W_1 \mid \{X_0=2\} \sim \mathcal{E}(1)$, $W_2 \mid \{X_{J_1}=3\} \sim \mathcal{E}(3)$

$\Rightarrow P(J_1 < 10 < J_2 \mid X_{J_1}=3, X_0=2) = \frac{1}{4}\left(e^{-10} - e^{-3 \cdot 10}\right) \approx 0.00001135$

c) Lanac je ireducibilan i regularan pa je stacionarna dist. ujedno i granična.

stac. dist: $\lambda Q = 0$, $\lambda_1+\lambda_2+\lambda_3=1$

$$\begin{cases} -2\lambda_1 + 3/2\,\lambda_3 = 0 & \Rightarrow \lambda_1 = \frac{3}{4}\lambda_3 \\ 2\lambda_1 - \lambda_2 + 3/2\,\lambda_3 = 0 & \\ \lambda_2 - 3\lambda_3 = 0 & \Rightarrow \lambda_2 = 3\lambda_3 \end{cases}$$

$\Rightarrow \lambda = \left(\frac{3}{4}\lambda_3, 3\lambda_3, \lambda_3\right)$, a zbog

$\lambda_1+\lambda_2+\lambda_3 = 1 \Rightarrow \lambda_3 = \frac{4}{19}$

$\Rightarrow$ granična dist. je

$\lambda = \left(\frac{3}{19}, \frac{12}{19}, \frac{4}{19}\right)$ //

d) Traži se $\frac{1}{m_3 q(3)} = \lambda_3 = \frac{4}{19}$ $\Rightarrow$ lanac provede asimptotski očekivano $\approx 21\%$ vremena u stanju 3!

e) Traži se $m_3 = \frac{1}{\lambda_3 q(3)} = \frac{1}{\frac{4}{19}\cdot 3} \approx 1.5833$ // (očekivano vrijeme povratka u stanje 3 je 1.5833 dana!

gdje je $m_3 = E\left[\inf\{t \geq J_1 : X_t = i\} \mid X_0 = i\right]$

4. Koristimo supstituciju $f(t, X_t) = \log X_t$

Prema Itôvoj Lemi II imamo:

$$f(t,X_t) - f(0,X_0) = \int_0^t \left(f_1(s,X_s) - \frac{\theta \gamma X_s}{\beta - 1} f_2(s,X_s) + \frac{\theta X_s^2}{\beta-1} f_{22}(s,X_s) \right) ds + \int_0^t \sqrt{\frac{2\theta}{\beta-1}}\, X_s f_2(s,X_s)\, dB_s$$

$$= \int_0^t \left(0 - \frac{\theta\gamma}{\beta-1} \cancel{X_s} \cdot \frac{1}{\cancel{X_s}} + \frac{\theta \cancel{X_s^2}}{\beta-1} \cdot \left(-\frac{1}{\cancel{X_s^2}}\right)\right) ds + \int_0^t \sqrt{\frac{2\theta}{\beta-1}}\, \cancel{X_s}\, \frac{1}{\cancel{X_s}}\, dB_s$$

$$= \int_0^t \left(-\frac{\theta\gamma}{\beta-1} - \frac{\theta}{\beta-1}\right) ds + \int_0^t \sqrt{\frac{2\theta}{\beta-1}}\, dB_s = -\frac{\theta(\gamma+1)}{\beta-1} t + \sqrt{\frac{2\theta}{\beta-1}}\, B_t$$

$$\Rightarrow X_t = 3\, e^{-\frac{\theta(\gamma+1)}{\beta-1} t + \sqrt{\frac{2\theta}{\beta-1}} B_t}$$ //

$\Rightarrow$ rješenje je jako jer je SDJ linearna...