Numerička integracija

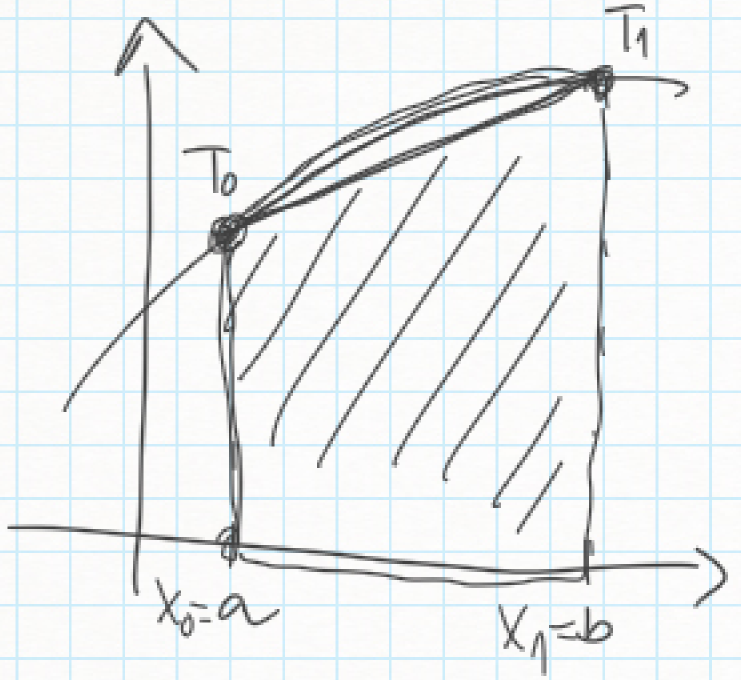
Imamo zadanu funkciju $f:[a,b]\to\mathbb{R}$, f neprekidna.

Problem: izračunati $\underbrace{I=\int_a^b f(x)\,dx}$

→ razlozi problema:

- primitivnu funkciju nije moguće dobiti elementarnim metodama
- podintegralnu funkciju poznamo u nekoliko točaka

TRAPEZNO PRAVILO



$$I=\int_a^b f(x)\,dx$$

f-ju f interpoliramo linearnom funkcijom P_1 u čvorovima $x_0=a$ i $x_1=b$

$$T_0(a,f(a)),\ T_1(b,f(b)) \Rightarrow \underbrace{P_1(x)=f(x)+\frac{f(b)-f(a)}{b-a}(x-a)}$$

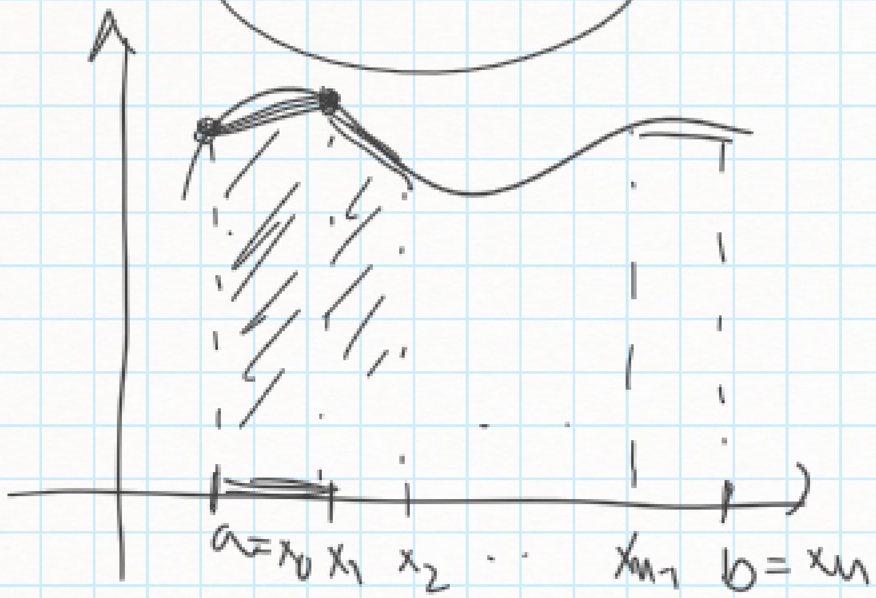
Tada imamo $\underline{\underline{I}}^* = \int_a^b P_1(x)\,dx = \frac{b-a}{2}\left(f(a)+f(b)\right)$

→ trapez sa stranicama $f(a)$ i $f(b)$ i visinom $v = b-a$.

Radi bolje aproksimacije podijelit ćemo interval $[a,b]$ na više podintervala.

Neka su $x_0, \dots, x_n \in [a,b]$, $x_1 - x_0 = x_2 - x_1 = \dots = x_n - x_{n-1} = h$; $x_0 = a$, $x_n = b$

$h = \frac{b-a}{n}$, $x_i = a + ih$, $i = 0, 1, \dots, n$, $y_i = f(x_i)$



* na svakom podintervalu primijenimo trapezno pravilo

$\Rightarrow I^* = \frac{h}{2}\left(y_0 + 2\cdot\sum_{i=1}^{n-1} y_i + y_n\right)$

PRODUŽENA (GENERALIZIRANA) TRAPEZNA FORMULA

$$\underline{I} = \underline{I}^* + E_n , \quad E_n = -\frac{b-a}{12} h^2 \cdot f''(c)$$

$$\Delta \underline{I}^* = |E_n| \leq \frac{b-a}{12} \cdot h^2 M_2 \quad \text{gdje je} \quad M_2 = \max_{a \leq x \leq b} |f''(x)|$$

Primjedba: Ako je zadana točnost $\varepsilon > 0$, možemo odrediti potreban broj podintervala da bi se postigla točnost ε:

$$\Delta \underline{I}^* \leq \frac{b-a}{12} h^2 M_2 < \varepsilon$$

$$\frac{b-a}{12} \cdot \frac{(b-a)^2}{n^2} \cdot M_2 < \varepsilon \quad \Big/ \cdot \frac{n^2}{\varepsilon}$$

$$n^2 > \frac{b-a}{12} (b-a)^2 \cdot \frac{1}{\varepsilon} M_2$$

$$\boxed{n > b-a \cdot \sqrt{\frac{b-a}{12} \cdot \frac{M_2}{\varepsilon}}}$$

Zad1 Izračunajte integral $\int_0^1 \sqrt{1+2x}\,dx$

$\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C$

a) Koristeći trapeznu formulu.

b) Koristeći produljenu trapeznu formulu (uz točnost $\varepsilon = 0.0005$)

Rj.

* izračunajmo egzaktnu vrijednost integrala (to se ovdje može napraviti)

$$I = \int_0^1 \sqrt{1+2x}\,dx = \left| \begin{array}{c} 1+2x = t \;/' \\ 2dx = dt \;/ :2 \\ dx = \frac{dt}{2} \\ x=0 \Rightarrow t=1 \\ x=1 \Rightarrow t=3 \end{array} \right| = \int_1^3 \sqrt{t} \cdot \frac{dt}{2} = \frac{1}{2} \int_1^3 t^{\frac{1}{2}}\,dt$$

$3^{\frac{3}{2}} = \sqrt{3^3} = 3 \cdot \sqrt{3}$

$$= \frac{1}{2} \cdot \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \Big|_1^3 = \frac{1}{2} \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} \Big|_1^3 = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} t^{\frac{3}{2}} \Big|_1^3 = \frac{1}{3} t^{\frac{3}{2}} \Big|_1^3 = \frac{1}{3}\left(3^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) =$$

$$= \frac{1}{3}(3 \cdot \sqrt{3} - 1)$$

$$= 1.39872$$

$f(x) = \sqrt{1+2x}$

a) $\underline{I}^* = \frac{f(a)+f(b)}{2} \cdot (b-a) = \frac{f(0)+f(1)}{2}(1-0) = \frac{1}{2} \cdot (1+\sqrt{3}) \approx \underline{\underline{1.36603}}$

$a = 0$

$b = 1$

b) $\underline{\varepsilon = 0.0005}$

$n = ?$

$\frac{b-a}{12} \cdot h^2 M_2 < \varepsilon$

$\underline{n > (b-a) \sqrt{\frac{b-a}{12} \cdot \frac{M_2}{\varepsilon}}} \to ?$

$a = 0, \quad b = 1$

$M_2 = \max_{x \in [0,1]} |f''(x)| = ?$

$f(x) = \sqrt{1+2x} = (1+2x)^{\frac{1}{2}}$

$f'(x) = \frac{1}{\cancel{2}} (1+2x)^{-\frac{1}{2}} \cancel{2} = \frac{1}{(1+2x)^{\frac{1}{2}}} = \underline{(1+2x)^{-\frac{1}{2}}}$

$f''(x) = -\frac{1}{\cancel{2}} (1+2x)^{-\frac{3}{2}} \cdot \cancel{2} = -\frac{1}{(1+2x)^{\frac{3}{2}}} = -\frac{1}{\sqrt{(1+2x)}}$

za $x \in [0,1]$ $\quad 1+2x>0 \Rightarrow (1+2x)^3 > 0 \,/\sqrt{\ } \Rightarrow \sqrt{(1+2x)^3} > 0 \Rightarrow \dfrac{1}{\sqrt{(1+2x)^3}} > 0 \,/ \cdot (-1)$

$$f''(x) = -\frac{1}{\sqrt{(1+2x)^3}} < 0$$

$$M_2 = \max_{x\in[0,1]} |f''(x)| = \max_{x\in[0,1]} \Bigg| \underbrace{-\frac{1}{\sqrt{(1+2x)^3}}}_{<0} \Bigg| = \max_{x\in[0,1]} \underbrace{\frac{1}{\sqrt{(1+2x)^3}}}_{\text{pada}} = \frac{1}{\sqrt{(1+2\cdot 0)^3}} = 1$$

$$\underset{\in \mathbb{N}}{n} > (b-a)\cdot\sqrt{\frac{b-a}{12}\cdot\frac{M_2}{\varepsilon}} = 1\cdot\sqrt{\frac{1}{12}\cdot\frac{1}{0.0005}} \approx 12.90994$$

$\Rightarrow n = 13$

$h = \dfrac{b-a}{n} = \dfrac{1}{13}$ $\quad x_i = a + ih$, $i = 0,1,\dots,13$, $y_i = f(x_i) = \sqrt{1+2x_i}$

$x_i = 0 + ih$

$x_i = i \cdot \frac{1}{13}$

x_i	y_i
0	1
1/13	1.07417
2/13	1.14354
3/13	1.20894
4/13	1.27098
5/13	1.33012
6/13	1.38675
7/13	1.44115
8/13	1.49358
9/13	1.54422

x_i	y_i
10/13	1.59326
11/13	1.64083
12/13	1.68705
1	1.73205

$$I^* = \frac{h}{2}\left(y_0 + 2\cdot\sum_{i=1}^{12} y_i + y_{13}\right) = 1.39851$$

Zadatak 2 Izračunajte potreban broj podintervala da bi se primjenom generalizirane trapezne formule odredila približna vrijednost integrala $I = \int_1^2 e^{-x^2}\,dx$ uz točnost $\varepsilon = 0.003$. Primjenom generalizirane trapezne formule odredite približnu vrijednost uz danu točnost.

Rj.

$n = ?$

$$n > (b-a)\cdot\sqrt{\frac{b-a}{12}\cdot\frac{M_2}{\varepsilon}}$$

$a=1,\ b=2$

$\varepsilon = 0.003$

$$M_2 = \max_{x\in[1,2]} |f''(x)|$$

$$f(x) = e^{-x^2}$$

$$f'(x) = e^{-x^2}\cdot(-x^2)' = -2x\cdot e^{-x^2}$$

$$f''(x) = -2\cdot e^{-x^2} + (-2x)\cdot(-2x\cdot e^{-x^2})$$

$$= -2\cdot e^{-x^2} + 4x^2\cdot e^{-x^2} = e^{-x^2}(4x^2-2)$$

$$f'''(x) = (-2x\cdot e^{-x^2})\cdot(4x^2-2) + e^{-x^2}\cdot(8x)$$

$$= e^{-x^2}(-8x^3+4x+8x) =$$

$f'''(x) = \underbrace{e^{-x^2}}_{>0}(12x - 8x^3)$

$24 - 8 \cdot 8$

$$\boxed{f''(x) = e^{-x^2} \cdot (4x^2 - 2)}$$

$f'''(x) = 0$

$\underbrace{e^{-x^2}}_{\neq 0} \cdot (12x - 8x^3) = 0$

$12x - 8x^3 = 0$

$x(12 - 8x^2) = 0$

$\underbrace{x = 0}_{\notin [1,2]}, \quad 12 - 8x^2 = 0$

$x_{1,2} = \pm\sqrt{\frac{3}{2}}$

$x_2 = -\sqrt{\frac{3}{2}} \notin [1,2]$

$\rightarrow$ kandidati za max $\sqrt{\frac{3}{2}}, 1, 2$

$f''(1) = 0.735, \quad f''(2) = 0.25, \quad f''\left(\sqrt{\frac{3}{2}}\right) = 0.8925$

$$\underline{\underline{M_2}} = \max_{x\in[1,2]} \left| \underbrace{e^{-x^2} \cdot (4x^2-2)}_{>0} \right| = \max_{x\in[1,2]} \underbrace{e^{-x^2} \cdot (4x^2-2)}_{f^4(x)} = \underline{\underline{0.8925}}$$

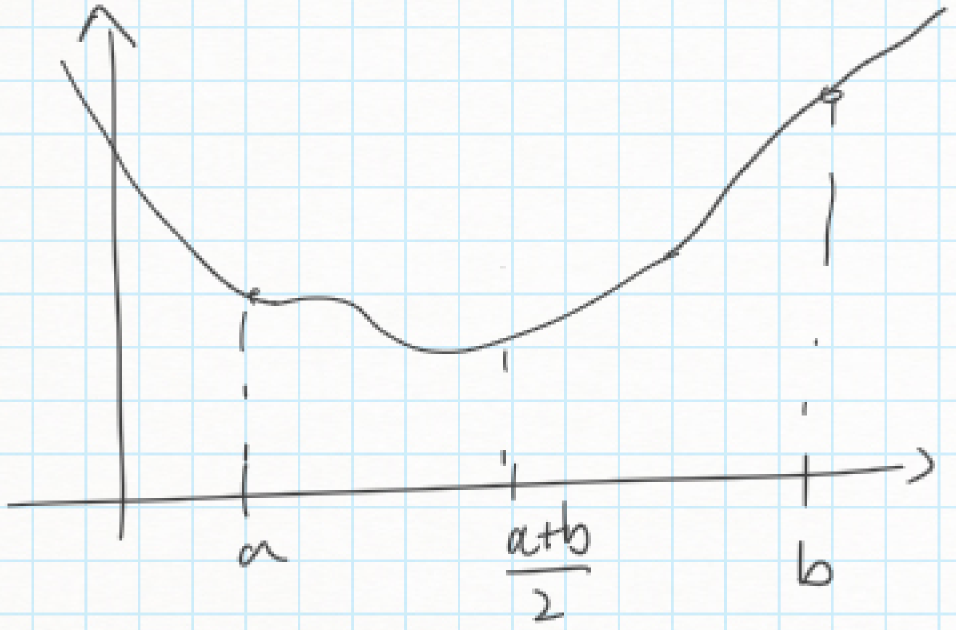
$$n > (b-a) \cdot \sqrt{\frac{b-a}{12} \cdot \frac{M^2}{\varepsilon}} = 4.979 \Rightarrow n = 5$$

$$h = \frac{b-a}{n} = \frac{2-1}{5} = \frac{1}{5} \quad , \; x_i = 1 + i \cdot \frac{1}{5} \quad , \; i = 0, 1, \ldots, 5 \; , \; y_i = e^{-x_i^2}$$

x_i	y_i
1	0.36788
1.2	0.23693
1.4	0.14086
1.6	0.0773
1.8	0.03916
2	0.01832

$$I^* = \frac{h}{2}\left(y_0 + 2 \cdot \sum_{i=1}^{4} y_i + y_5\right) = \underline{\underline{0.13747}}$$

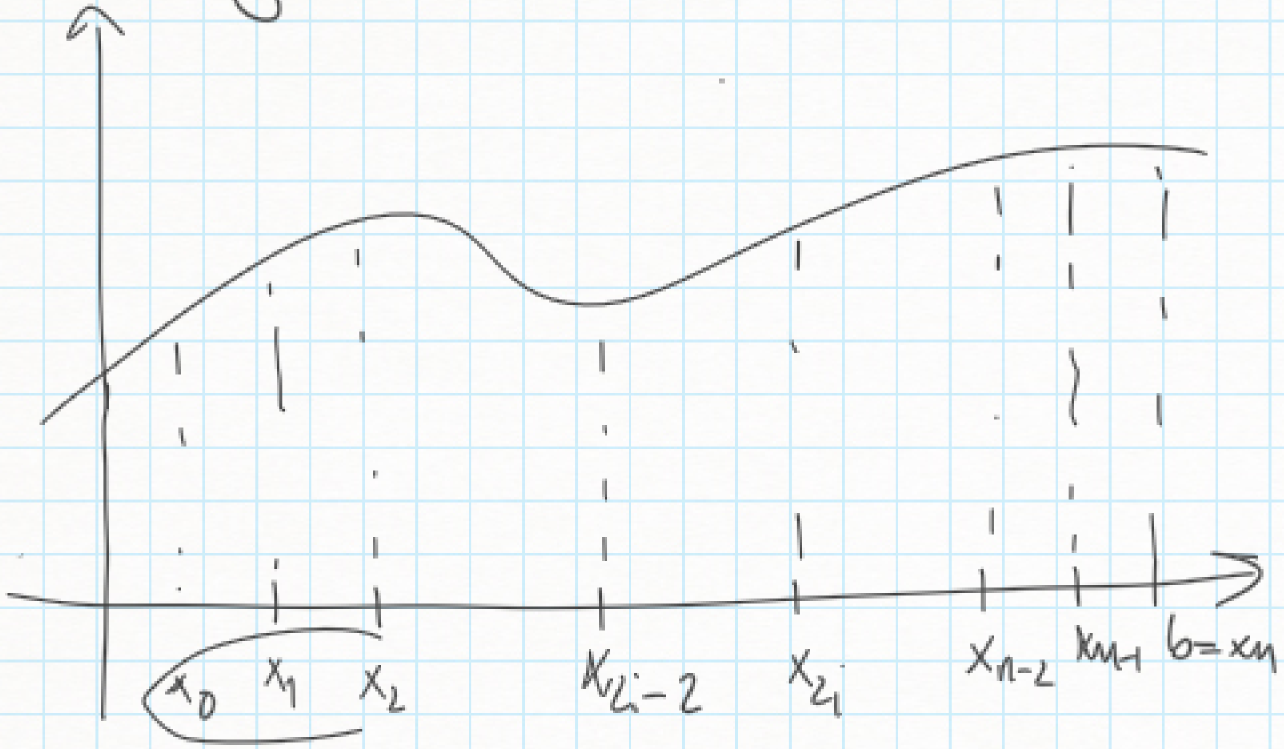
SIMPSONOVA FORMULA



→ kroz tri točke povučemo parabolu pa integriramo

$$\boxed{I^* = \frac{b-a}{6}\left(f(a) + 4\cdot f\left(\frac{a+b}{2}\right) + f(b)\right)}$$

PRODUŽENA SIMPSONOVA FORMULA



→ $[a,b]$ podijelimo na parni broj intervala $n=2m$

$h = \frac{b-a}{n}$, $x_i = a + ih$, $i=0,1,\dots,n$, $y_i = f(x_i)$

$$I^* = \frac{h}{3}\left(y_0 + y_{2m} + 2\sum_{i=1}^{m-1} y_{2i} + 4\cdot\sum_{i=1}^{m} y_{2i-1}\right)$$

Primjedba: Koliko podintervala treba „napraviti" da bi se zadovoljila točnost $\varepsilon > 0$?

$$I = I^* + E_n$$

$$|E_n| = \frac{b-a}{180} \cdot h^4 \cdot |f^{(4)}(c)| \quad , \quad c \in (a,b)$$

$$|I - I^*| \leq \frac{b-a}{180} h^4 \cdot M_4 \quad , \quad M_4 = \max_{x \in [a,b]} |f^{(4)}(x)|$$

$$\frac{b-a}{180} \cdot h^4 \cdot M_4 < \varepsilon$$

$$n > \sqrt[4]{\frac{(b-a)^5}{180} \cdot \frac{M_4}{\varepsilon}} \Rightarrow \boxed{n > (b-a)\sqrt[4]{\frac{b-a}{180} \cdot \frac{M_4}{\varepsilon}}}$$

NAPOMENA: n mora biti paran!

Zad3 Pomoću Simpsonove formule izračunajte $\int_0^1 \sqrt{1+2x}\,dx$.

Rj. $f(x) = \sqrt{1+2x}$

$a=0$, $b=1$

$$I^* = \frac{b-a}{6}\left(f(a) + 4\cdot f\left(\frac{a+b}{2}\right) + f(b)\right) = \frac{1}{6}\cdot\left(1 + 4\cdot f\left(\left(\frac{1}{2}\right)\right) + \sqrt{3}\right) =$$

$$= \frac{1}{6}\cdot\left(1 + 4\cdot\sqrt{2} + \sqrt{3}\right) = 1.39815$$

Zad 4. Produženom Simpsonovom formulom odredite približnu vrijednost integrala

$$I = \int_0^1 \sqrt{1+2x}\, dx$$ na tri decimale točno.

Rj.

$$\varepsilon = 0.\underbrace{000}_{\text{tri decimale točne}}5$$

$$n > (b-a)\cdot \sqrt[4]{\frac{b-a}{180}\cdot\frac{M_4}{\varepsilon}}$$

$$M_4 = \max_{x\in[a,b]} |f^{(4)}(x)|$$

$a = 0$

$b = 1$

$$f(x) = \sqrt{1+2x} = (1+2x)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{\sqrt{1+2x}}$$

$$f''(x) = -\frac{1}{\sqrt{(1+2x)^3}} = -(1+2x)^{-\frac{3}{2}}$$

$$f'''(x) = \frac{3}{\cancel{2}}(1+2x)^{-\frac{5}{2}}\cdot\cancel{2} = 3\cdot(1+2x)^{-\frac{5}{2}}$$

$$f^{(4)}(x) = 3\cdot\left(-\frac{5}{\cancel{2}}\right)(1+2x)^{-\frac{7}{2}}\cdot\cancel{2} = \underbrace{-15\,\underbrace{(1+2x)^{-\frac{7}{2}}}_{>0}}_{<0}$$

$$M_4 = \max_{x\in[0,1]} \left| -15 \cdot \frac{1}{\sqrt{(1+2x)^7}} \right| = \max_{x\in[0,1]} 15 \cdot \frac{1}{\sqrt{(1+2x)^7}} \underset{x=0}{=} 15 \cdot \frac{1}{\sqrt{(1+0)^7}} = 15$$

$$< 0$$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$n > 1 \cdot \sqrt[4]{\frac{1}{180} \cdot \frac{15}{0.0005}} = 3.59$$

$$\Rightarrow \underline{\underline{n = 4}}$$

$$n = 2 \cdot m \Rightarrow m = 2 \quad , \quad h = \frac{1}{4}$$

$$x_i = a + ih \quad , i = 0,1,2,3,4$$

$$x_i = 0 + ih = i \cdot \frac{1}{4}$$

$$y_i = f(x_i) = \sqrt{1+2x_i}$$

x_i	y_i
0	1
1/4	1.2247
2/4	1.4142
3/4	1.5811
1	1.7321

$$I^* = \frac{4}{3}\left(y_0 + y_4 + 2 \cdot \sum_{i=1}^{1} y_{2i} + 4 \cdot \sum_{i=1}^{2} y_{2i-1}\right)$$

$$= \frac{1}{12}\left(\underline{y_0 + y_4 + 2 \cdot y_2 + 4 \cdot (y_1 + y_3)}\right)$$

$$= \frac{1}{12} \cdot (16.7837) = \underline{\underline{1.39864}}$$

Zad5 Izračunajte potreban broj podintervala da bi se odredila približna vrijednost integrala $I = \int_1^3 x^2 \ln x \, dx$ uz točnost $\varepsilon = 0.005$.

a) primjenom generalizirane trapezne formule

b) primjenom generalizirane Simpsonove formule.

Primjenom generalizirane Simpsonove formule odredite približnu vrijednost za zadanu točnost $\varepsilon = 0.005$. D.Z.

Rj.

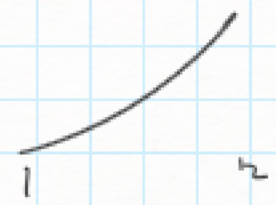
a) $n > (b-a) \cdot \sqrt{\frac{b-a}{12} \cdot \frac{M_2}{\varepsilon}}$, $M_2 = \max_{x \in [a,b]} |f''(x)|$, $a = 1$, $b = 3$

$f(x) = x^2 \cdot \ln x$

$f'(x) = 2x \cdot \ln x + x^2 \cdot \frac{1}{x} = 2x \cdot \ln x + x = x(2\ln x + 1)$

$f''(x) = 2 \ln x + 1 + \cancel{x} \cdot \left(2 \cdot \frac{1}{\cancel{x}}\right) = 2 \ln x + 3$ → $\ln x$ rastuća funkcija

rastuća



$$M_2 = \max_{x\in[1,3]} |f''(x)| = \max_{x\in[1,3]} |\underbrace{2\ln x + 3}_{\geq 0}| = \max_{x\in[1,3]} (\underbrace{2\ln x + 3}_{\text{rosnąca}}) = 2\cdot \ln 3 + 3 = 5.1972$$

$$n > 2\cdot\sqrt{\frac{2}{12}\cdot\frac{5.1972}{0.005}} = 26.32 \Rightarrow \underline{\underline{n = 27}}$$

b) $$n > (b-a)\sqrt[4]{\frac{b-a}{180}\cdot\frac{M_4}{\varepsilon}} = 2\cdot\sqrt[4]{\frac{2}{180}\cdot\frac{2}{0.005}} = 2.904 \Rightarrow n = 4,\ m = 2$$

$$f'''(x) = \frac{2}{x},\quad f^{(4)} = -\frac{2}{x^2}$$

$$M_4 = \max_{x\in[1,3]} \Big|\underbrace{-\frac{2}{x^2}}_{<0}\Big| = \max_{x\in[1,3]} \Big(\underbrace{\frac{2}{x^2}}_{\text{padająca}}\Big) = \frac{2}{1^2} = 2$$