The third International School on
Model Reduction for Dynamical Control Systems
October 5–10, 2015
Dubrovnik, Croatia

Model Order Reduction via System Balancing

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Outline

Outline

Introduction
Model Reduction for Dynamical Systems

Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & x(t_0) = x_0, \\ y(t) = g(t, x(t), u(t)) \end{cases}$$

with

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
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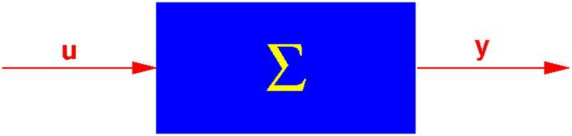

Model Reduction for Dynamical Systems

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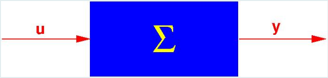

Reduced-Order Model (ROM)

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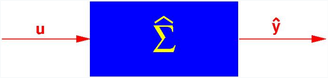

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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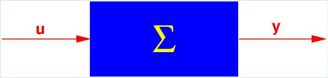

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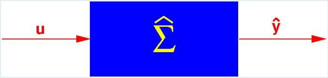

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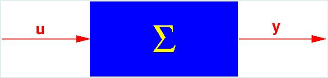

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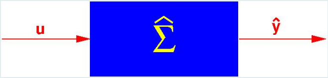

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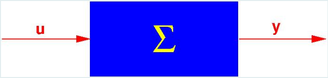

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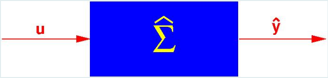


Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Secondary goal: reconstruct approximation of x from $\hat{x}$.

Model Reduction for Dynamical Systems

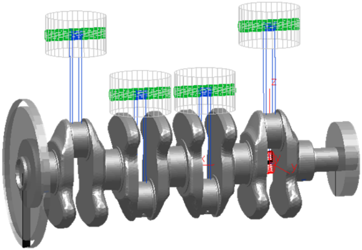
Linear Systems

Linear, Time-Invariant (LTI) Systems

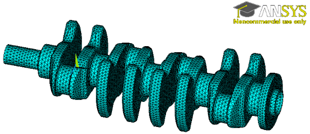
$$
\begin{array}{rclcll}
\dot{x} & = & f(t,x,u) & = & Ax + Bu, \quad A \in \mathbb{R}^{n\times n}, & B \in \mathbb{R}^{n\times m}, \\
y & = & g(t,x,u) & = & Cx + Du, \quad C \in \mathbb{R}^{q\times n}, & D \in \mathbb{R}^{q\times m}.
\end{array}
$$

Application Areas

Structural Mechanics / Finite Element Modeling **since ∼1960ies**



$\rightsquigarrow$



- Resolving complex 3D geometries ⇒ millions of degrees of freedom.
- Analysis of elastic deformations requires many simulation runs for varying external forces, in particular if the model is used in an (elastic) multi-body simulation ((E)MBS).

Standard MOR techniques in structural mechanics: modal truncation, combined with Guyan reduction (static condensation) $\rightsquigarrow$ Craig-Bampton method.

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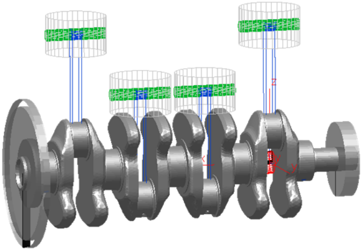

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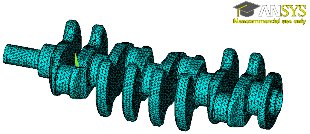

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(Optimal) Control

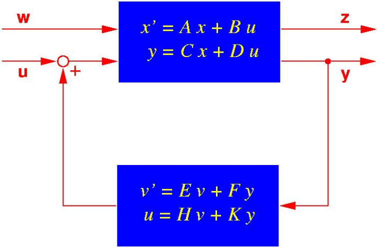
since ∼1980ies

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input = output of plant,
- output = input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$.



Practical controllers require small N ($N \sim 10$, say) due to

- real-time constraints,
- increasing fragility for larger N.

$\Longrightarrow$ reduce order of plant (n) and/or controller (N).

Standard MOR techniques in systems and control: balanced truncation and related methods.

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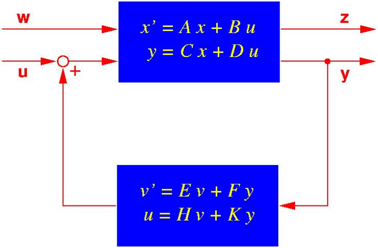

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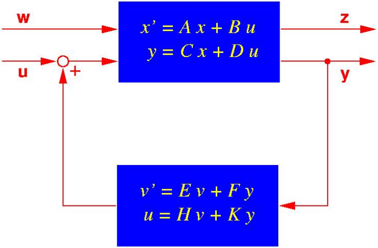

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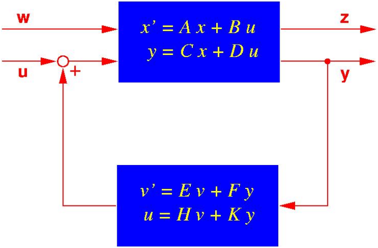

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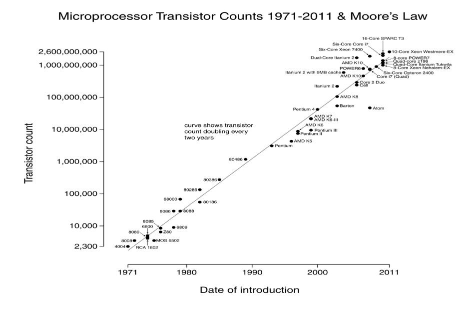
Micro Electronics/Circuit Simulation

since ∼1990ies

Progressive miniaturization

- Verification of VLSI/ULSI chip design requires high number of simulations for different input signals.
- **Moore's Law (1965/75)** states that the number of on-chip transistors doubles each 24 months.



Source: http://en.wikipedia.org/wiki/File:Transistor_Count_and_Moore'sLaw_-_2011.svg

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- Increase in packing density and multilayer technology requires modeling of interconncet to ensure that thermic/electro-magnetic effects do not disturb signal transmission.

Intel 4004 (1971)	Intel Core 2 Extreme (quad-core) (2007)
1 layer, 10μ technology	9 layers, $45nm$ technology
2,300 transistors	$> 8,200,000$ transistors
64 kHz clock speed	> 3 GHz clock speed.

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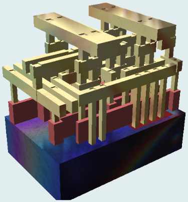

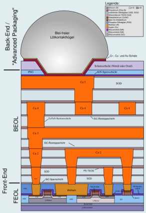

Source: http://en.wikipedia.org/wiki/Image:Silicon_chip_3d.png.

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- Here: mostly MOR for linear systems, they occur in micro electronics through modified nodal analysis (MNA) for RLC networks. e.g., when
  - decoupling large linear subcircuits,
  - modeling transmission lines,
  - modeling pin packages in VLSI chips,
  - modeling circuit elements described by Maxwell's equation using partial element equivalent circuits (PEEC).

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Standard MOR techniques in circuit simulation:
Krylov subspace / Padé approximation / rational interpolation methods.

Application Areas

Many other disciplines in computational sciences and engineering like

- computational fluid dynamics (CFD),
- computational electromagnetics,
- chemical process engineering,
- design of MEMS/NEMS (micro/nano-electrical-mechanical systems),
- computational acoustics,
- . . .
- **Current trend:** more and more multi-physics problems, i.e., coupling of several field equations, e.g.,
  - electro-thermal (e.g., bondwire heating in chip design),
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Peter Benner and Lihong Feng.
Model Order Reduction for Coupled Problems
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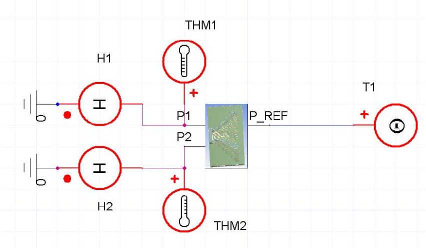
Motivating Examples

Electro-Thermic Simulation of Integrated Circuit (IC)

[Source: Evgenii Rudnyi, CADFEM GmbH]

- SIMPLORER® test circuit with 2 transistors.



- Conservative thermic sub-system in SIMPLORER:
  voltage ⇝ temperature, current ⇝ heat flow.
- Original model: $n = 270.593$, $m = q = 2 \Rightarrow$
  Computing time (on Intel Xeon dualcore 3GHz, 1 Thread):
  - Main computational cost for set-up data $\approx 22min$.
  - Computation of reduced models from set-up data: 44–49sec. ($r = 20$–70).
  - Bode plot (MATLAB on Intel Core i7, 2,67GHz, 12GB):
    7.5h for original system , $<$ 1min for reduced system.

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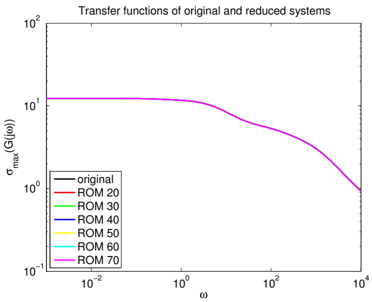
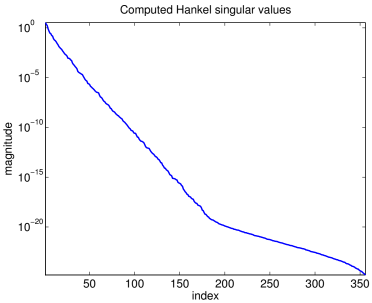
Bode Plot (Amplitude)



Hankel Singular Values



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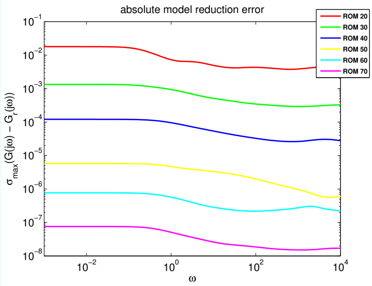
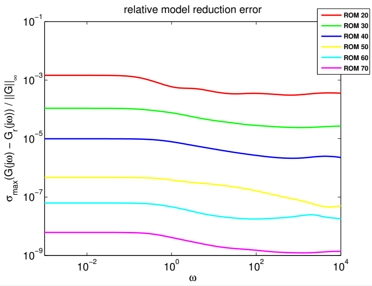
Absolute Error



Relative Error



Motivating Examples

A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System

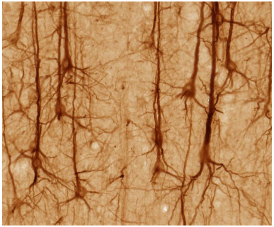
- Simple model for neuron (de-)activation [Chaturantabut/Sorensen 2009]

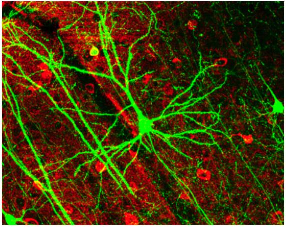
$$\epsilon v_t(x,t) = \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + g,$$
$$w_t(x,t) = hv(x,t) - \gamma w(x,t) + g,$$

with $f(v) = v(v-0.1)(1-v)$ and initial and boundary conditions

$$\begin{aligned} v(x,0) &= 0, & w(x,0) &= 0, & x &\in [0,1] \\ v_x(0,t) &= -i_0(t), & v_x(1,t) &= 0, & t &\geq 0, \end{aligned}$$

where $\epsilon = 0.015, h = 0.5, \gamma = 2, g = 0.05, i_0(t) = 50000t^3 \exp(-15t)$.





Source: http://en.wikipedia.org/wiki/Neuron

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where $\epsilon = 0.015, h = 0.5, \gamma = 2, g = 0.05, i_0(t) = 50000t^3 \exp(-15t)$.

- Parameter g handled as an additional input.
- Original state dimension $n = 2 \cdot 400$, QBDAE dimension $N = 3 \cdot 400$, reduced QBDAE dimension $r = 26$, chosen expansion point $\sigma = 1$.

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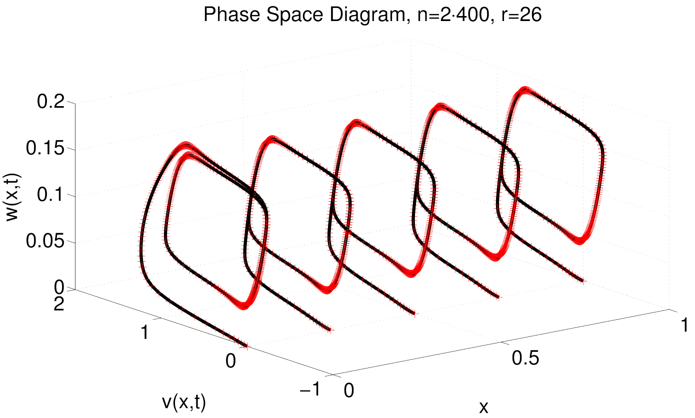

Outline

Numerical Linear Algebra

Image Compression by Truncated SVD

- A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where x_{ij} contains color information of pixel (i, j).
- Memory (in single precision): $4 \cdot n_x \cdot n_y$ bytes.

Theorem (Schmidt-Mirsky/Eckart-Young)

Best rank-r approximation to $X \in \mathbb{R}^{n_x \times n_y}$ w.r.t. spectral norm:

$$\widehat{X} = \sum_{j=1}^{r} \sigma_j u_j v_j^T,$$

where $X = U\Sigma V^T$ is the singular value decomposition (SVD) of X.
The approximation error is $\|X - \widehat{X}\|_2 = \sigma_{r+1}$.

Idea for dimension reduction

Instead of X save $u_1, \ldots, u_r, \sigma_1 v_1, \ldots, \sigma_r v_r$.
$\rightsquigarrow$ memory $= 4r \times (n_x + n_y)$ bytes.

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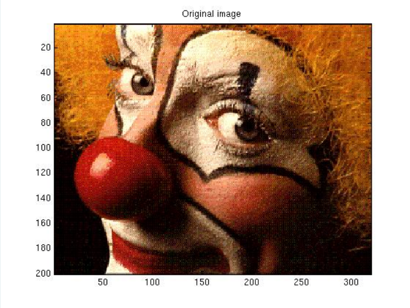
Example: Image Compression by Truncated SVD

Example: Clown



320×200 pixel
$\rightsquigarrow \quad \approx 256$ kB

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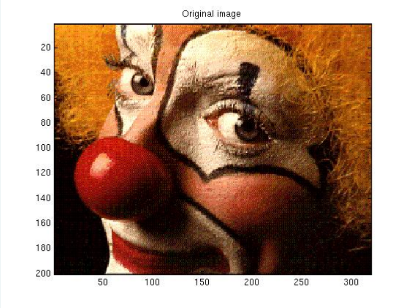

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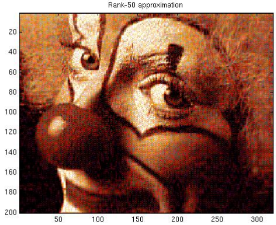
- rank $r = 50$, ≈ 104 kB



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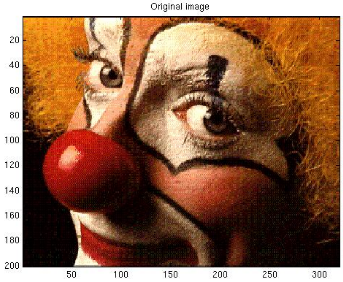

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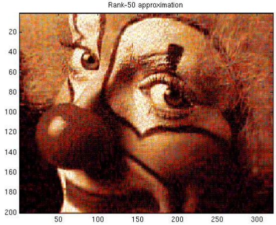
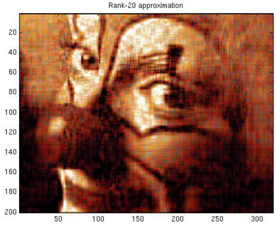


- rank $r = 20$, ≈ 42 kB



Dimension Reduction via SVD

Example: Gatlinburg

Organizing committee
Gatlinburg/Householder Meeting 1964:

James H. Wilkinson, Wallace Givens,

George Forsythe, Alston Householder,

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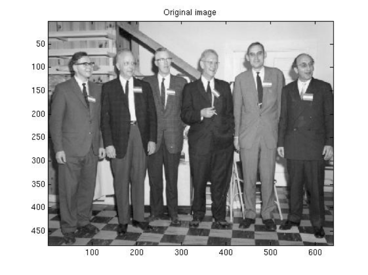

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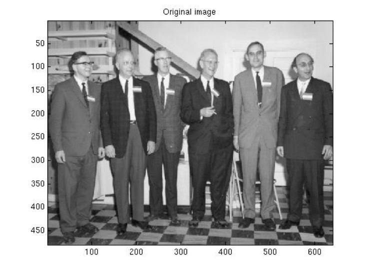
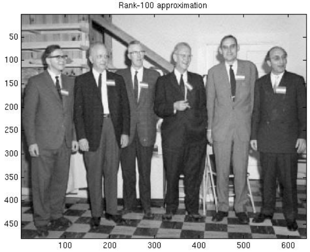
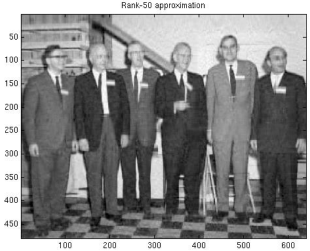


640 × 480 pixel, ≈ 1229 kB

- rank $r = 100$, ≈ 448 kB



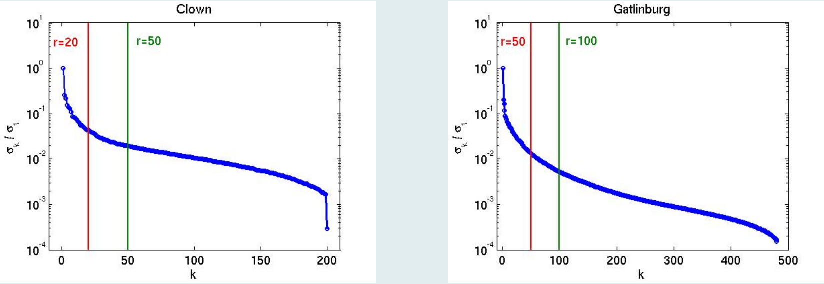
- rank $r = 50$, ≈ 224 kB



Background: Singular Value Decay

Image data compression via SVD works, if the singular values decay (exponentially).

Singular Values of the Image Data Matrices



A different viewpoint

Linear Mapping

A matrix $A \in \mathbb{R}^{\ell \times k}$ represents a linear mapping

$$\mathcal{A} : \mathbb{R}^k \to \mathbb{R}^\ell \ : \ x \to y := Ax.$$

The truncated SVD ignores small singular values and thus the related left and right singular vectors.

Consequence:

- Vectors (almost) in the kernel of A do not contribute to $\mathrm{range}\,(A)$ and can hardly or not at all be reconstructed from the input-output relation ("A^{-1}") $\rightsquigarrow$ "unobservable" states.
- Vectors (almost) in $\mathrm{range}\,(A)^\perp$ cannot be "reached" from any $x \in \mathbb{R}^k$ $\rightsquigarrow$ "unreachable/uncontrollable" states.
- Hence, the truncated SVD ignores states hard to reconstruct and hard to reach.

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Systems and Control Theory
The Laplace transform

Definition

The Laplace transform of a time domain function $f \in L_{1,\mathrm{loc}}$ with $\mathrm{dom}\,(f) = \mathbb{R}_0^+$ is

$$\mathcal{L} : f \mapsto F, \quad F(s) := \mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)\, dt, \quad s \in \mathbb{C}.$$

F is a function in the (Laplace or) frequency domain.

Note: for frequency domain evaluations ("frequency response analysis"), one takes $\mathrm{re}\, s = 0$ and $\mathrm{im}\, s \geq 0$. Then $\omega := \mathrm{im}\, s$ takes the role of a frequency (in [rad/s], i.e., $\omega = 2\pi v$ with v measured in [Hz]).

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Lemma

$$\mathcal{L}\{\dot{f}(t)\}(s) = sF(s) - f(0).$$

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Note: for ease of notation, in the following we will use lower-case letters for both, a function and its Laplace transform!

Systems and Control Theory

The Model Reduction Problem as Approximation Problem in Frequency Domain

Linear Systems in Frequency Domain

Application of Laplace transform $(x(t) \mapsto x(s),\ \dot{x}(t) \mapsto sx(s))$ to linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

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$$y(s) = \Big(\underbrace{C(sI_n - A)^{-1}B + D}_{=:G(s)}\Big)u(s).$$

$G(s)$ is the transfer function of Σ.

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Goal: Fast evaluation of mapping $u \rightarrow y$.

Systems and Control Theory

The Model Reduction Problem as Approximation Problem in Frequency Domain

Formulating model reduction in frequency domain

Approximate the dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y &= Cx + Du, \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}, \end{aligned}$$

by reduced-order system

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \quad \hat{A} \in \mathbb{R}^{r\times r},\ \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, \quad \hat{C} \in \mathbb{R}^{q\times r},\ \hat{D} \in \mathbb{R}^{q\times m} \end{aligned}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\|.$$

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of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\|.$$

$\Longrightarrow$ Approximation problem: $\min\limits_{\text{order}\,(\hat{G})\leq r} \|G - \hat{G}\|.$

Systems and Control Theory
Properties of linear systems

Definition

A linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

is stable if its transfer function $G(s)$ has all its poles in the left half plane and it is asymptotically (or Lyapunov or exponentially) stable if all poles are in the open left half plane $\mathbb{C}^- := \{z \in \mathbb{C} \,|\, \Re(z) < 0\}$.

Lemma

Sufficient for asymptotic stability is that A is asymptotically stable (or Hurwitz), i.e., the spectrum of A, denoted by $\Lambda(A)$, satisfies $\Lambda(A) \subset \mathbb{C}^-$.

Note that by abuse of notation, often *stable system* is used for asymptotically stable systems.

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Questions:

- For fixed $x_0 \in \mathbb{R}^n$ and some $x^1 \in \mathbb{R}^n$, is there a feasible control function $u \in \mathcal{U}_{ad}$ and time $t_1 > t_0 = 0$ such that $x(t_1; u) = x^1$?
  What is the set of targets x^1 reachable from x^0?
- For fixed $x_1 \in \mathbb{R}^n$ and some $x^0 \in \mathbb{R}^n$, is there a feasible control function $u \in \mathcal{U}_{ad}$ and time $t_1 > t_0 = 0$ such that $x(t_1; u) = x^1$?
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E.g., $\mathcal{U}_{ad} \in \{C^k[0, T], L_2(0, T)\}$, possibly with constraints $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$.

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Systems and Control Theory
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Definition (Controllability)

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c) If the system is controllable to x^1 for all $x^0 \in \mathbb{R}^n$, it is (completely) controllable.

The controllability set w.r.t. x^1 is defined as $\mathcal{C} := \bigcup_{t_1>0} \mathcal{C}(t_1)$ where

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$$\mathcal{C}(t_1) := \{x^0 \in \mathbb{R}^n; \exists u \in \mathcal{U}_{ad} : x(t_1; u) = x^1\}.$$

In short: an LTI system is controllable $\Longleftrightarrow \mathcal{C} = \mathbb{R}^n$.

E.g., $\mathcal{U}_{ad} \in \{C^k[0, T], L_2(0, T)\}$, possibly with constraints $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$.

Systems and Control Theory
Properties of linear systems

Now: characterize controllability.

Systems and Control Theory
Properties of linear systems

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Variation of constants $\Longrightarrow$

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds = e^{At}(x^0 + \int_0^t e^{-As}Bu(s)ds).$$

Systems and Control Theory

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Hence, if x^0 is controllable to x^1:

$$x^1 = x(t_1) = e^{At_1}x^0 + \int_0^{t_1} e^{A(t_1-t)}Bu(t)dt$$

This is equivalent to

$$e^{-At_1}x^1 - x^0 = \int_0^{t_1} e^{-At}Bu(t)dt.$$

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Ansatz: $u(t) = B^T e^{-A^T t}c \Longrightarrow$

$$e^{-At_1}x^1 - x^0 = \int_0^{t_1} e^{-At}BB^T e^{-A^T t}dt c =: P(0, t_1)c.$$

Systems and Control Theory

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Hence, an LTI system is controllable iff this linear system is solvable for $c \in \mathbb{R}^n$, i.e., iff $P(0, t_1)$ is invertible. (Note: $P(0, t_1) = P(0, t_1)^T \geq 0$ by definition!)

Systems and Control Theory

Properties of linear systems

Now: characterize controllability.

Theorem

For an LTI system defined by $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, *T.F.A.E.:*

a) *The LTI system* $\dot{x} = Ax + Bu$ *is controllable.*

b) *The finite time Gramian* $P(0, t_1)$ *is spd* $\forall\, t_1 > 0$.

c) *The controllability matrix*

$$K(A, B) := [B, AB, A^2B, \ldots, A^{n-1}B] \in \mathbb{R}^{n\times n\cdot m}$$

has full rank n. (Note: $\mathrm{range}\,(K(A, B)) = \mathcal{C}(t_1)\ \forall\, t_1 > 0$*!)*

d) *If z is a left eigenvector of A, then* $z^*B \neq 0$.

e) *(Hautus test)* $\mathrm{rank}([\lambda I - A, B]) = n\ \forall \lambda \in \mathbb{C}$.

Systems and Control Theory

Properties of linear systems

The Gramian characterization of controllability for stable systems can be based on positive definiteness of the (infinite) controllability Gramian

$$P := \int_0^\infty e^{As} B B^T e^{A^T s} ds,$$

using congruence of $P(0, t_1)$ to $\int_0^{t_1} e^{As} B B^T e^{A^T s} ds$ and taking the limit $t_1 \to \infty$.

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using congruence of $P(0, t_1)$ to $\int\limits_0^{t_1} e^{As} BB^T e^{A^T s} ds$ and taking the limit $t_1 \to \infty$.

Theorem

For a stable LTI system defined by $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$, T.F.A.E.:

a) *The LTI system $\dot{x} = Ax + Bu$ is controllable.*

b) *The controllability Gramian P is positive definite.*

Systems and Control Theory

Properties of linear systems

New question: suppose we have

$$y(t) = \tilde{y}(t)$$

corresponding to two trajectories $x, \tilde{x}$ obtained by the same input function $u(t)$. Can we conclude that $x(0) = \tilde{x}(0)$, or even stronger, that $x(t) = \tilde{x}(t)$ for $t \leq 0, t \geq 0$ (past/future)?
(Note that $x(t_0) = \tilde{x}(t_0)$ is sufficient as trajectory uniquely determined. In other words, is the mapping $x^0 \to y(t)$ injective?)

Systems and Control Theory
Properties of linear systems

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Definition (Observability)

An LTI system is reconstructable (observable) if for solution trajectories $x(t), \tilde{x}(t)$ obtained with the same input function u, we have

$$\begin{aligned} & y(t) = && \tilde{y}(t) \quad \forall t \le 0 \quad (\forall t \ge 0) \\ \Longrightarrow \quad & x(t) = && \tilde{x}(t) \quad \forall t \le 0 \quad (\forall t \ge 0). \end{aligned}$$

Systems and Control Theory

Properties of linear systems

Characterization of observability/reconstructability:

Theorem (Duality)

An LTI system is reconstructable if and only if the dual system $\dot{x}(t) = -A^T x(t) - C^T u(t)$ is controllable.

Systems and Control Theory
Properties of linear systems

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Theorem

For an LTI system defined by $(A, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{q \times n}$, *T.F.A.E.:*

a) *The LTI system is reconstructable.*

b) *The LTI system is observable.*

c) *The observability matrix*

$$\mathcal{O}(A, C) = \left[C^T, A^T C^T, (A^2)^T C, \ldots, (A^{n-1})^T C^T \right]^T \in \mathbb{R}^{np \times n} \text{ has rank } n.$$

d) *If* $Ax = \lambda x$, *then* $C^T x \neq 0$.

e) *(Hautus test)* $\operatorname{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$.

Systems and Control Theory
Properties of linear systems

Characterization of observability/reconstructability:

Theorem (Duality)

An LTI system is reconstructable if and only if the dual system $\dot{x}(t) = -A^T x(t) - C^T u(t)$ *is controllable.*

Theorem

A stable LTI system is observable if and only if the observability Gramian

$$Q := \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

is symmetric positive definite.

Systems and Control Theory

Properties of linear systems

- Controllability/observability are sometimes too strong.
- Weaker requirement: is there $u \in \mathcal{U}_{ad}$ to steer x_0 to vicinity of x^1?
- For LTI systems, it suffices to consider $x^1 = 0$!
- Hence, is there $u \in \mathcal{U}_{ad}$ so that $\lim_{t\to\infty} x(t; u) = 0$ ($\forall\, x^0 \in \mathbb{R}^n$)?
- If the answer is **yes**, then the LTI system is called stabilizable

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- If the answer is **yes**, then the LTI system is called stabilizable

Theorem

For an LTI system defined by $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$*, T.F.A.E.:*

a) *The LTI system is stabilizable.*

b) $\exists F \in \mathbb{R}^{m\times n}$ *with* $\Lambda(A + BF) \subset \mathbb{C}^-$.

c) *If* $p^* A = \tilde{\lambda} p^*$ *and* $\mathrm{Re}(\lambda) \geq 0$, *then* $p^* B \neq 0$.

d) $\mathrm{rank}([A - \lambda I, B]) = n \quad \forall \lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda) \geq 0$.

e) *In the (controllability) Kalman decomposition of* (A, B),

$$V^T A V = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, V^T B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

we have $\Lambda(A_3) \subset \mathbb{C}^-$.

Systems and Control Theory

Properties of linear systems

$\exists$ dual concept of stabilizability, analogous to duality of controllability and observability.

Definition (Detectability)

An LTI system is detectable if for any solution $x(t)$ of $\dot{x} = Ax$ with $Cx(t) \equiv 0$ we have $\lim_{t\to\infty} x(t) = 0$.

(We can not observe all of x, but the unobservable part is stable.)

Systems and Control Theory
Properties of linear systems

$\exists$ dual concept of stabilizability, analogous to duality of controllability and observability.

Theorem

For an LTI system defined by $(A, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{q\times n}$, T.F.A.E.:

a) *The LTI system is detectable.*

b) (A^T, C^T) *is stabilizable.*

c) $Ax = \lambda x, \mathrm{Re}(\lambda) \geq 0 \Rightarrow C^T x \neq 0.$

d) $\mathrm{rank}\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ *for all* $\lambda, \mathrm{Re}(\lambda) \geq 0.$

e) *In the observability Kalman decomposition of* (A^T, C^T),

$$W^T AW = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}, CW = [C_1 \; 0],$$

we have $\Lambda(A_3) \subset \mathbb{C}^-$.

Systems and Control Theory
Realizations of Linear Systems

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \qquad \begin{array}{l} \text{with transfer function} \\ G(s) = C(sI - A)^{-1}B + D, \end{array}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is called a realization of Σ.

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Realizations are not unique!

Transfer function is invariant under state-space transformations,

$$\mathcal{T} : \left\{ \begin{array}{ccc} x & \to & Tx, \\ (A, B, C, D) & \to & (TAT^{-1}, TB, CT^{-1}, D), \end{array} \right.$$

Systems and Control Theory
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the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is called a realization of Σ.

Realizations are not unique!

Transfer function is invariant under addition of uncontrollable/unobservable states:

$$\frac{d}{dt}\begin{bmatrix} x \\ x_1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + \begin{bmatrix} B \\ B_1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + Du(t),$$

$$\frac{d}{dt}\begin{bmatrix} x \\ x_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & C_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + Du(t),$$

for arbitrary $A_j \in \mathbb{R}^{n_j\times n_j}$, $j = 1, 2$, $B_1 \in \mathbb{R}^{n_1\times m}$, $C_2 \in \mathbb{R}^{q\times n_2}$ and any $n_1, n_2 \in \mathbb{N}$.

Systems and Control Theory

Realizations of Linear Systems

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the quadruple $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$ is called a realization of Σ.

Realizations are not unique!

Hence,

$$(A, B, C, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}, \begin{bmatrix} B \\ B_1 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D \right),$$

$$(TAT^{-1}, TB, CT^{-1}, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & C_2 \end{bmatrix}, D \right),$$

are all realizations of Σ!

Systems and Control Theory
Realizations of Linear Systems

Definition

For a linear (time-invariant) system

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the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is called a realization of Σ.

Definition

The McMillan degree of Σ is the unique minimal number $\hat{n} \geq 0$ of states necessary to describe the input-output behavior completely.
A minimal realization is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order $\hat{n}$.

Systems and Control Theory
Realizations of Linear Systems

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Definition

The McMillan degree of Σ is the unique minimal number $\hat{n} \geq 0$ of states necessary to describe the input-output behavior completely.
A minimal realization is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order $\hat{n}$.

Theorem

A realization (A, B, C, D) of a linear system is minimal $\iff$ (A, B) is controllable and (A, C) is observable.

Systems and Control Theory
Balanced Realizations

Definition

A realization (A, B, C, D) of a linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \mathrm{diag}\{\sigma_1, \dots, \sigma_n\} \quad (\text{w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \dots, n-1).$$

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When does a balanced realization exist?

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When does a balanced realization exist?
Assume A to be Hurwitz, i.e. $\Lambda(A) \subset \mathbb{C}^-$. Then:

Theorem

Given a stable minimal linear system $\Sigma : (A, B, C, D)$, a balanced realization is obtained by the state-space transformation with

$$T_b := \Sigma^{-\frac{1}{2}} V^T R,$$

where $P = S^T S$, $Q = R^T R$ (e.g., Cholesky decompositions) and $SR^T = U \Sigma V^T$ is the SVD of SR^T.

Proof. Easy.

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$\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of Σ.

Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

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Theorem

The infinite controllability/observability Gramians P/Q satisfy the Lyapunov equations

$$AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0.$$

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Proof. Exercise!

Systems and Control Theory

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$\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of Σ.

Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

Theorem

The Hankel singular values (HSVs) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!

Systems and Control Theory
Balanced Realizations

Theorem

The Hankel singular values (HSVs) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!

Proof. In balanced coordinates, the HSVs are $\Lambda(PQ)^{\frac{1}{2}}$. Now let

$$(\hat{A}, \hat{B}, \hat{C}, D) = (TAT^{-1}, TB, CT^{-1}, D)$$

be any transformed realization with associated controllability Lyapunov equation

$$0 = \hat{A}\hat{P} + \hat{P}\hat{A}^T + \hat{B}\hat{B}^T = TAT^{-1}\hat{P} + \hat{P}T^{-T}A^T T^T + TBB^T T^T.$$

This is equivalent to

$$0 = A(T^{-1}\hat{P}T^{-T}) + (T^{-1}\hat{P}T^{-T})A^T + BB^T.$$

The uniqueness of the solution of the Lyapunov equation implies that $\hat{P} = TPT^T$ and, analogously, $\hat{Q} = T^{-T}QT^{-1}$. Therefore,

$$\hat{P}\hat{Q} = TPQT^{-1},$$

showing that $\Lambda(\hat{P}\hat{Q}) = \Lambda(PQ) = \{\sigma_1^2, \ldots, \sigma_n^2\}$.

Systems and Control Theory

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$$P = Q = \operatorname{diag}\{\sigma_1, \ldots, \sigma_n\} \quad (\text{w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1).$$

$\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of Σ.

Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

Remark

For non-minimal systems, the Gramians can also be transformed into diagonal matrices with the leading $\hat{n} \times \hat{n}$ submatrices equal to $\operatorname{diag}(\sigma_1, \ldots, \sigma_{\hat{n}})$, and

$$\hat{P}\hat{Q} = \operatorname{diag}(\sigma_1^2, \ldots, \sigma_{\hat{n}}^2, 0, \ldots, 0).$$

see [Laub/Heath/Paige/Ward 1987, Tombs/Postlethwaite 1987].

Qualitative and Quantitative Study of the Approximation Error
System Norms

Consider transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and input functions $u \in \mathcal{L}_2^m \cong L_2^m(-\infty, \infty)$, with the L_2-norm

$$\|u\|_2^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} u(\jmath\omega)^H u(\jmath\omega)\, d\omega.$$

Assume A (asympotically) stable: $\Lambda\,(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} \,:\, \operatorname{re} z < 0\}$.

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$$\int_{-\infty}^{\infty} y(\jmath\omega)^H y(\jmath\omega)\, d\omega \quad = \quad \int_{-\infty}^{\infty} u(\jmath\omega)^H G(\jmath\omega)^H G(\jmath\omega) u(\jmath\omega)\, d\omega$$

(Here, $\|\,.\,\|$ denotes the Euclidian vector or spectral matrix norm.)

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$$\Longrightarrow y \in \mathcal{L}_2^q \cong L_2^q(-\infty, \infty).$$

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Consequently, the 2-induced operator norm

$$\|G\|_\infty := \sup_{\|u\|_2 \neq 0} \frac{\|Gu\|_2}{\|u\|_2}$$

is well defined. It can be shown that

$$\|G\|_\infty = \sup_{\omega\in\mathbb{R}} \|G(\jmath\omega)\| = \sup_{\omega\in\mathbb{R}} \sigma_{\max}\,(G(\jmath\omega))\,.$$

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Sketch of proof:
$\|G(\jmath\omega)u(\jmath\omega)\| \leq \|G(\jmath\omega)\|\|u(\jmath\omega)\| \Rightarrow$ "$\leq$".
Construct u with $\|Gu\|_2 = \sup_{\omega \in \mathbb{R}} \|G(\jmath\omega)\|\|u\|_2$.

Qualitative and Quantitative Study of the Approximation Error

System Norms

Consider transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D.$$

Hardy space $\mathcal{H}_\infty$

Function space of matrix-/scalar-valued functions that are analytic and bounded in $\mathbb{C}^+$.

The $\mathcal{H}_\infty$-norm is

$$\|F\|_\infty := \sup_{\operatorname{re} s > 0} \sigma_{\max}\left(F(s)\right) = \sup_{\omega \in \mathbb{R}} \sigma_{\max}\left(F(\jmath\omega)\right).$$

Stable transfer functions are in the Hardy spaces

- $\mathcal{H}_\infty$ in the SISO case (single-input, single-output, $m = q = 1$);
- $\mathcal{H}_\infty^{q \times m}$ in the MIMO case (multi-input, multi-output, $m > 1, q > 1$).

Qualitative and Quantitative Study of the Approximation Error
System Norms

Consider transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D.$$

Consequence of Parseval identity/Plancherel Theorem

$$L_2(-\infty,\infty) \cong \mathcal{L}_2, \quad L_2(0,\infty) \cong \mathcal{H}_2$$

Consequently, 2-norms in time and frequency domains coincide!

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$\mathcal{H}_\infty$ approximation error

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}$.

$$\|y - \hat{y}\|_2 = \|Gu - \hat{G}u\|_2 \leq \|G - \hat{G}\|_\infty \|u\|_2.$$

$\Longrightarrow$ compute reduced-order model such that $\|G - \hat{G}\|_\infty < tol$!

Note: error bound holds in time- and frequency domain due to Plancherel!

Qualitative and Quantitative Study of the Approximation Error
System Norms

Consider stable transfer function

$$G(s) = C\,(sI - A)^{-1}\,B, \quad \text{i.e. } D = 0.$$

Hardy space $\mathcal{H}_2$

Function space of matrix-/scalar-valued functions that are analytic $\mathbb{C}^+$ and bounded w.r.t. the $\mathcal{H}_2$-norm

$$\begin{aligned} \|F\|_2 &:= \frac{1}{2\pi}\left(\sup_{\operatorname{re}\sigma>0}\int_{-\infty}^{\infty}\|F(\sigma+\jmath\omega)\|_F^2\,d\omega\right)^{\frac{1}{2}} \\ &= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty}\|F(\jmath\omega)\|_F^2\,d\omega\right)^{\frac{1}{2}}. \end{aligned}$$

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$\mathcal{H}_2$ approximation error for impulse response $(u(t) = u_0\delta(t))$

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B}$.

$$\|y - \hat{y}\|_2 = \|Gu_0\delta - \hat{G}u_0\delta\|_2 \leq \|G - \hat{G}\|_2\|u_0\|.$$

$\Longrightarrow$ compute reduced-order model such that $\|G - \hat{G}\|_2 < tol$!

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Theorem (Practical Computation of the $\mathcal{H}_2$-norm)

$$\|F\|_2^2 = \operatorname{tr}\left(B^T Q B\right) = \operatorname{tr}\left(C P C^T\right),$$

where P, Q are the controllability and observability Gramians of the corresponding LTI system.

Qualitative and Quantitative Study of the Approximation Error
Approximation Problems

Output errors in time-domain

$$
\begin{aligned}
\|y-\hat{y}\|_2 &\leq \|G-\hat{G}\|_\infty \|u\|_2 &&\Longrightarrow \|G-\hat{G}\|_\infty < \mathrm{tol} \\
\|y-\hat{y}\|_\infty &\leq \|G-\hat{G}\|_2 \|u\|_2 &&\Longrightarrow \|G-\hat{G}\|_2 < \mathrm{tol}
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Qualitative and Quantitative Study of the Approximation Error

Approximation Problems

Output errors in time-domain

$$\begin{aligned} \|y-\hat{y}\|_2 &\leq \|G-\hat{G}\|_\infty \|u\|_2 && \Longrightarrow \|G-\hat{G}\|_\infty < \mathrm{tol} \\ \|y-\hat{y}\|_\infty &\leq \|G-\hat{G}\|_2 \|u\|_2 && \Longrightarrow \|G-\hat{G}\|_2 < \mathrm{tol} \end{aligned}$$

$\mathcal{H}_\infty$-norm	best approximation problem for given reduced order r in general open; balanced truncation yields suboptimal solution with computable $\mathcal{H}_\infty$-norm bound.
$\mathcal{H}_2$-norm	necessary conditions for best approximation known; (local) optimizer computable with iterative rational Krylov algorithm (IRKA)
Hankel-norm $\|G\|_H := \sigma_{\max}$	optimal Hankel norm approximation (AAK theory).

Qualitative and Quantitative Study of the Approximation Error

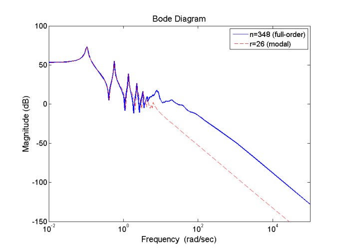
Computable error measures

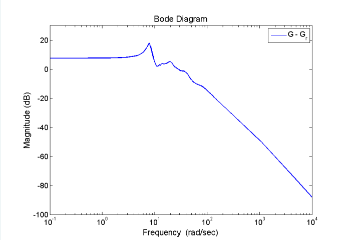
Evaluating system norms is computationally very (sometimes too) expensive.

Other measures

- absolute errors $\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_2$, $\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_\infty$ $(j = 1, \ldots, N_\omega)$;
- relative errors $\frac{\|G(\jmath\omega_j)-\hat{G}(\jmath\omega_j)\|_2}{\|G(\jmath\omega_j)\|_2}$, $\frac{\|G(\jmath\omega_j)-\hat{G}(\jmath\omega_j)\|_\infty}{\|G(\jmath\omega_j)\|_\infty}$;
- "eyeball norm", i.e. look at frequency response/Bode (magnitude) plot: for SISO system, log-log plot frequency vs. $|G(\jmath\omega)|$ (or $|G(\jmath\omega) - \hat{G}(\jmath\omega)|$) in decibels, 1 dB $\simeq 20\log_{10}$(value).

  For MIMO systems, $q \times m$ array of plots G_{ij}.





Outline

Model Reduction by Projection

Goals

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Preserve physical properties:
  - stability (poles of G in $\mathbb{C}^-$),
  - minimum phase (zeroes of G in $\mathbb{C}^-$),
  - passivity

$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

("system does not generate energy").

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Model Reduction by Projection

Projection Basics

Definition (Projector)

A projector is a matrix $P \in \mathbb{R}^{n\times n}$ with $P^2 = P$. Let $\mathcal{V} = \mathrm{range}\,(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\, v_1, \ldots, v_r \,]$, then $P = V(V^T V)^{-1} V^T$ is a projector onto $\mathcal{V}$.

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Lemma (Projector Properties)

- If $P = P^T$, then P is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector (aka: Petrov-Galerkin projection).
- P is the identity operator on $\mathcal{V}$, i.e., $Pv = v \;\; \forall v \in \mathcal{V}$.
- $I - P$ is the complementary projector onto $\ker P$.
- If $\mathcal{V}$ is an A-invariant subspace corresponding to a subset of A's spectrum, then we call P a spectral projector.
- Let $\mathcal{W} \subset \mathbb{R}^n$ be another r-dimensional subspace and $W = [\, w_1, \dots, w_r \,]$ be a basis matrix for $\mathcal{W}$, then $P = V(W^T V)^{-1} W^T$ is an oblique projector onto $\mathcal{V}$ along $\mathcal{W}$.

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- Let $\mathcal{W} \subset \mathbb{R}^n$ be another r-dimensional subspace and $W = [\, w_1, \ldots, w_r \,]$ be a basis matrix for $\mathcal{W}$, then $P = V(W^T V)^{-1} W^T$ is an oblique projector onto $\mathcal{V}$ along $\mathcal{W}$.

Model Reduction by Projection

Projection-based MOR Methods

Methods:

1. Modal Truncation
2. Balanced Truncation
3. Rational Interpolation (Padé-Approximation and (rational) Krylov Subspace Methods)
4. many more...

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!

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$$\mathrm{range}\,(V) = \mathcal{V}, \quad \mathrm{range}\,(W) = \mathcal{W}, \quad W^T V = I_r.$$

Then, with $\hat{x} = W^T x$, we obtain $x \approx V\hat{x}$ so that

$$\|x - \tilde{x}\| = \|x - V\hat{x}\|,$$

and the reduced-order model is

$$\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

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Important observation:

- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

$$W^T \left(\dot{\tilde{x}} - A\tilde{x} - Bu\right) \quad = \quad W^T \left(VW^T \dot{x} - AVW^T x - Bu\right)$$

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Outline

Modal Truncation

Basic method:

Assume A is diagonalizable, $T^{-1}AT = D_A$, project state-space onto A-invariant subspace $\mathcal{V} = \operatorname{span}(t_1, \ldots, t_r)$, t_k = eigenvectors corresp. to "dominant" modes / eigenvalues of A. Then with

$$V = T(:, 1:r) = [\, t_1, \ldots, t_r \,], \quad \tilde{W}^H = T^{-1}(1:r, :), \quad W = \tilde{W}(V^H\tilde{W})^{-1},$$

reduced-order model is

$$\hat{A} := W^H A V = \operatorname{diag}\{\lambda_1, \ldots, \lambda_r\}, \quad \hat{B} := W^H B, \quad \hat{C} = CV$$

Also computable by truncation:

$$T^{-1}AT = \begin{bmatrix} \hat{A} & \\ & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}, \quad CT = [\, \hat{C}, \; C_2 \,], \quad \hat{D} = D.$$

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Properties:

Simple computation for large-scale systems, using, e.g., Krylov subspace methods (Lanczos, Arnoldi), Jacobi-Davidson method.

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Properties:

Error bound:

$$\|G - \hat{G}\|_\infty \leq \|C_2\|\|B_2\| \frac{1}{\min_{\lambda \in \Lambda(A_2)} |\mathrm{Re}(\lambda)|}.$$

Proof:

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D = CTT^{-1}(sI - A)^{-1}TT^{-1}B + D \\ &= CT(sI - T^{-1}AT)^{-1}T^{-1}B + D \\ &= [\,\hat{C},\; C_2\,] \begin{bmatrix} (sI_r - \hat{A})^{-1} & \\ & (sI_{n-r} - A_2)^{-1} \end{bmatrix} \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix} + D \\ &= \hat{G}(s) + C_2(sI_{n-r} - A_2)^{-1}B_2, \end{aligned}$$

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$$G(s) \quad = \quad \hat{G}(s) + C_2(sI_{n-r} - A_2)^{-1}B_2,$$

observing that $\|G - \hat{G}\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(C_2(\jmath\omega I_{n-r} - A_2)^{-1}B_2)$, and

$$C_2(\jmath\omega I_{n-r} - A_2)^{-1}B_2 = C_2 \mathrm{diag}\left(\frac{1}{\jmath\omega - \lambda_{r+1}}, \ldots, \frac{1}{\jmath\omega - \lambda_n}\right) B_2.$$

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Difficulties:

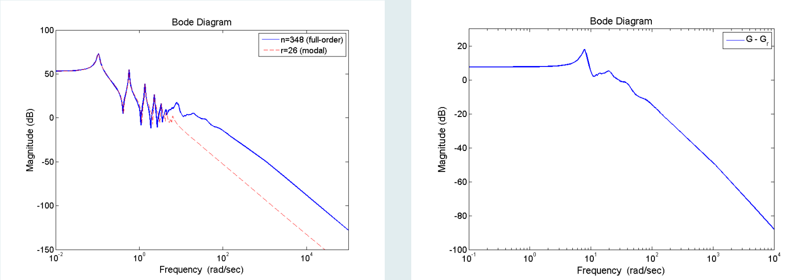
- Eigenvalues contain only limited system information.
- Dominance measures are difficult to compute.
  ([LITZ '79] use Jordan canoncial form; otherwise merely heuristic criteria, e.g., [VARGA '95]. Recent improvement: dominant pole algorithm.)
- Error bound not computable for really large-scale problems.

Basic Principle

Example

BEAM, SISO system from SLICOT Benchmark Collection for Model Reduction, $n = 348$, $m = q = 1$, reduced using 13 dominant complex conjugate eigenpairs, error bound yields $\|G - \hat{G}\|_\infty \leq 1.21 \cdot 10^3$

Bode plots of transfer functions and error function



MATLAB® demo.

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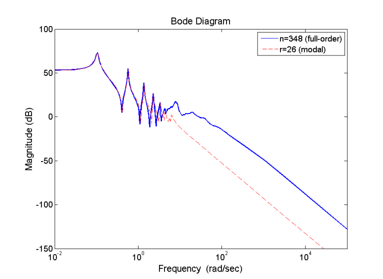

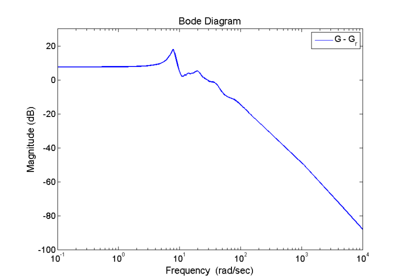

MATLAB® demo.

Basic Principle

Extensions

Base enrichment

Static modes are defined by setting $\dot{x} = 0$ and assuming unit loads, i.e., $u(t) \equiv e_j$, $j = 1, \ldots, m$:

$$0 = Ax(t) + Be_j \quad \Longrightarrow \quad x(t) \equiv -A^{-1} b_j.$$

Projection subspace $\mathcal{V}$ is then augmented by $A^{-1}[\, b_1, \ldots, b_m \,] = A^{-1}B$.

Interpolation-projection framework $\Longrightarrow G(0) = \hat{G}(0)$!

If two sided projection is used, complimentary subspace can be augmented by $A^{-T}C^T \Longrightarrow G'(0) = \hat{G}'(0)$!

Note: if $m \neq q$, add random vectors or delete some of the columns in $A^{-T}C^T$.

Basic Principle

Extensions

Guyan reduction (static condensation)

Partition states in masters $x_1 \in \mathbb{R}^r$ and slaves $x_2 \in \mathbb{R}^{n-r}$ (FEM terminology)
Assume stationarity, i.e., $\dot{x} = 0$ and solve for x_2 in

$$\begin{aligned} 0 &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ \Rightarrow \quad x_2 &= -A_{22}^{-1} A_{21} x_1 - A_{22}^{-1} B_2 u. \end{aligned}$$

Inserting this into the first part of the dynamic system

$$\dot{x}_1 = A_{11} x_1 + A_{12} x_2 + B_1 u, \quad y = C_1 x_1 + C_2 x_2$$

then yields the reduced-order model

$$\begin{aligned} \dot{x}_1 &= (A_{11} - A_{12} A_{22}^{-1} A_{21}) x_1 + (B_1 - A_{12} A_{22}^{-1} B_2) u \\ y &= (C_1 - C_2 A_{22}^{-1} A_{21}) x_1 - C_2 A_{22}^{-1} B_2 u. \end{aligned}$$

Modal Truncation

Dominant Pole Algorithm

Pole-Residue Form of Transfer Function

Consider partial fraction expansion of transfer function with $D = 0$:

$$G(s) = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k}$$

with the residues $R_k := (Cx_k)(y_k^H B) \in \mathbb{C}^{q \times m}$.

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Note: this follows using the spectral decomposition $A = XDX^{-1}$, with $X = [\, x_1, \dots, x_n]$ the right and $X^{-1} =: Y = [y_1, \dots, y_n]^H$ the left eigenvector matrices:

$$\begin{aligned}
G(s) &= C(sI - XDX^{-1})^{-1}B = CX(sI - \operatorname{diag}\{\lambda_1, \dots, \lambda_n\})^{-1}YB \\
&= [\, Cx_1, \dots, Cx_n \,] \begin{bmatrix} \frac{1}{s-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_n} \end{bmatrix} \begin{bmatrix} y_1^H B \\ \vdots \\ y_n^H B \end{bmatrix} \\
&= \sum_{k=1}^{n} \frac{(Cx_k)(y_k^H B)}{s - \lambda_k}.
\end{aligned}$$

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Note: $R_k = (Cx_k)(y_k^H B)$ are the residues of G in the sense of the residue theorem of complex analysis:

$$\operatorname{res}\,(G, \lambda_\ell) = \lim_{s \to \lambda_\ell} (s - \lambda_\ell) G(s) = \sum_{k=1}^{n} \underbrace{\lim_{s \to \lambda_\ell} \frac{s - \lambda_\ell}{s - \lambda_k}}_{= \begin{cases} 0 \text{ for } k \neq \ell \\ 1 \text{ for } k = \ell \end{cases}} R_k = R_\ell.$$

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As projection basis use spaces spanned by right/left eigenvectors corresponding to dominant poles, i.e.. $(\lambda_j,\ x_j,\ y_j)$ with largest

$$\|R_k\| / |\operatorname{re}(\lambda_k)|.$$

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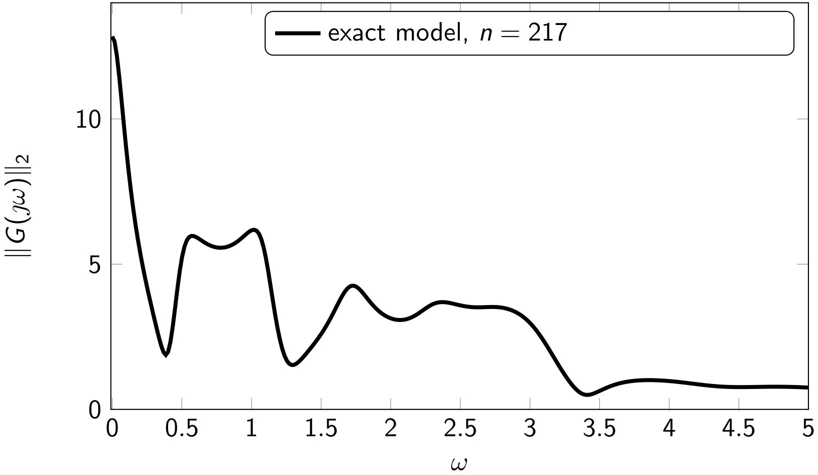
Remark

The dominant modes have most important influence on the input-output behavior of the system and are responsible for the "peaks"' in the frequency response.

Dominant Poles

Random SISO Example ($B,\ C^T \in \mathbb{R}^n$)



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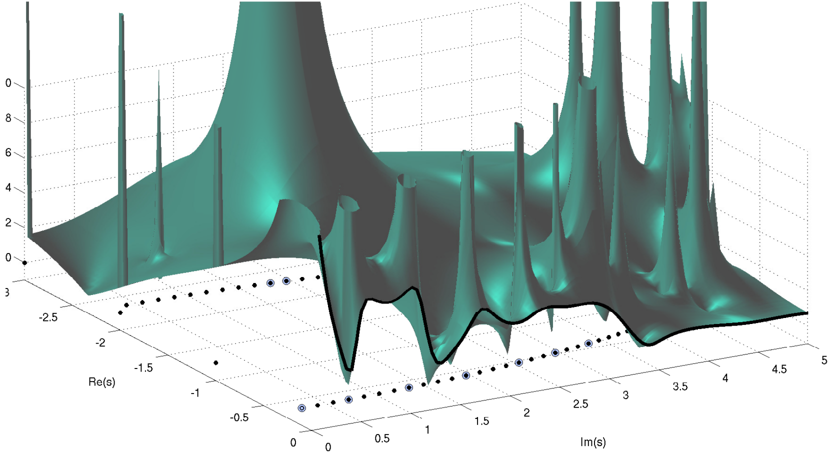

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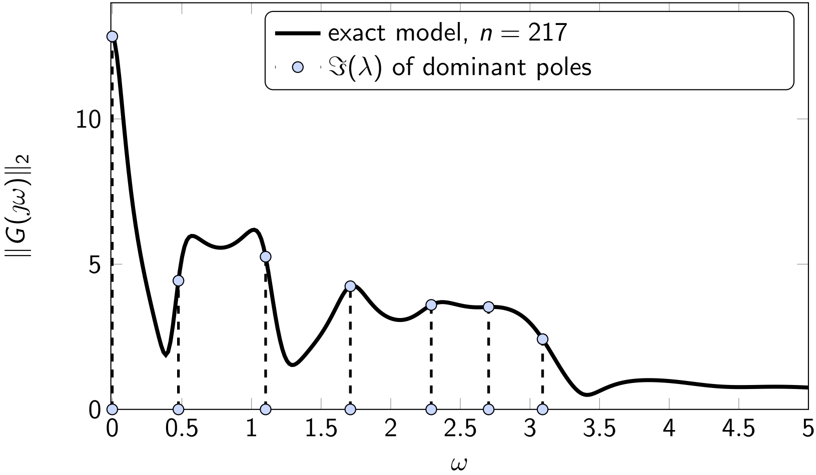

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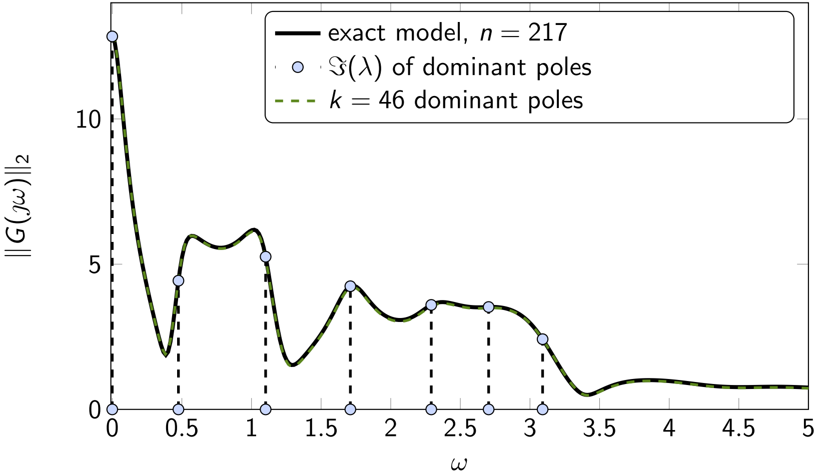

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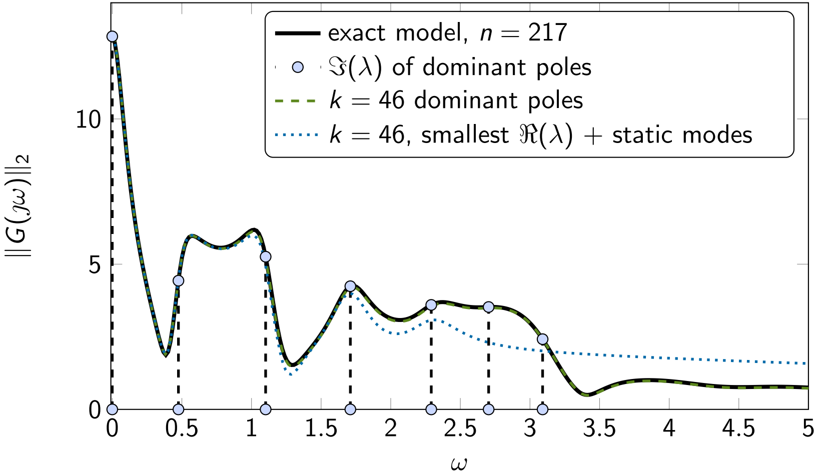
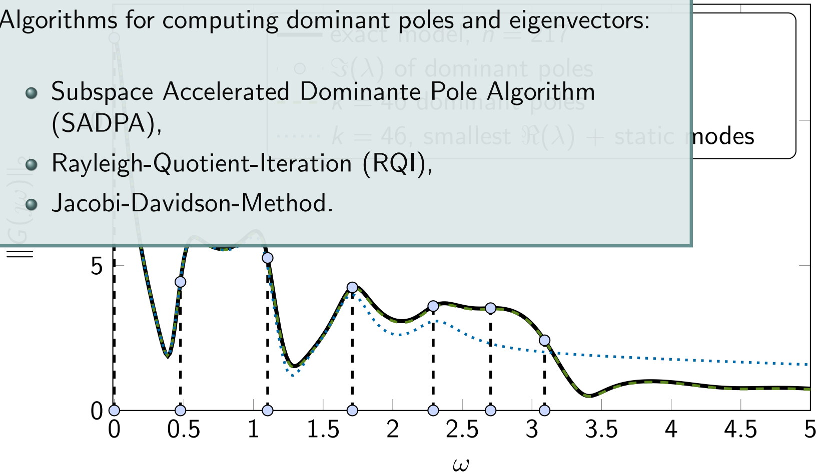


Dominant Poles

Random SISO Example (B, $C^T \in \mathbb{R}^n$)

Algorithms for computing dominant poles and eigenvectors:

- Subspace Accelerated Dominante Pole Algorithm (SADPA),
- Rayleigh-Quotient-Iteration (RQI),
- Jacobi-Davidson-Method.



Outline

Balanced Truncation

Basic principle:

- Recall: a stable system Σ, realized by (A, B, C, D), is called balanced, if the Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

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- Compute balanced realization of the system via state-space transformation

$$\begin{aligned}\mathcal{T} : (A, B, C, D) \quad &\mapsto \quad (TAT^{-1}, TB, CT^{-1}, D) \\ &= \quad \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D\right)\end{aligned}$$

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Balanced Truncation

Motivation:

The HSVs $\Lambda\left(PQ\right)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are system invariants: they are preserved under

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in transformed coordinates, the Gramians satisfy

$$\begin{aligned}
(TAT^{-1})(TPT^T) + (TPT^T)(TAT^{-1})^T + (TB)(TB)^T &= 0, \\
(TAT^{-1})^T(T^{-T}QT^{-1}) + (T^{-T}QT^{-1})(TAT^{-1}) + (CT^{-1})^T(CT^{-1}) &= 0
\end{aligned}$$

$$\Rightarrow\ (TPT^T)(T^{-T}QT^{-1}) = TPQT^{-1},$$

hence $\Lambda\,(PQ) = \Lambda\,((TPT^T)(T^{-T}QT^{-1}))$.

Balanced Truncation

Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, $P = S^T S$, $Q = R^T R$.
2. Compute SVD $SR^T = [\, U_1, \, U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$.
3. ROM is $(W^T AV, W^T B, CV, D)$, where

$$W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \qquad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.$$

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3. ROM is $(W^T AV, W^T B, CV, D)$, where

$$W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \qquad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.$$

Note:

$$V^T W \quad = \quad (\Sigma_1^{-\frac{1}{2}} U_1^T S)(R^T V_1 \Sigma_1^{-\frac{1}{2}})$$

Balanced Truncation

Implementation: SR Method

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$\Longrightarrow$ VW^T is an oblique projector, hence balanced truncation is a Petrov-Galerkin projection method.

Balanced Truncation

Properties:

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \leq \left(2 \sum\nolimits_{k=r+1}^{n} \sigma_k\right) \|u\|_2.$$

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Balanced Truncation

Theoretical Background

Linear, Time-Invariant (LTI) Systems

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\begin{aligned}
\dot{x} &= Ax + Bu, \qquad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \\
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Assumptions (for now): $t_0 = 0$, $x_0 = x(0) = 0$, $D = 0$.

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State-Space Description for I/O-Relation

Variation-of-constants $\Longrightarrow$

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Alternative to State-Space Operator: Hankel Operator

Instead of

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$$\mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t > 0.$$

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$\leadsto$ *Hankel singular values* $\{\sigma_j\}_{j=1}^{\infty} : \ \sigma_1 \geq \sigma_2 \geq \ldots \geq 0.$

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$\Longrightarrow$ SVD-type approximation of $\mathcal{H}$ possible!

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$\Downarrow$

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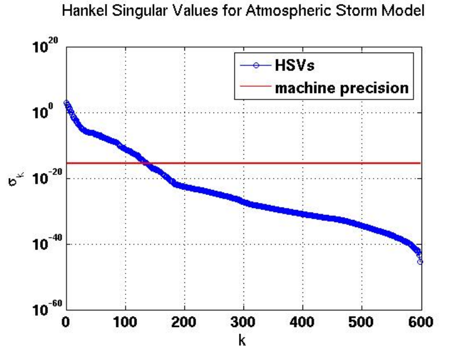

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$\Rightarrow$ solution: Adamjan-Arov-Krein (AAK Theory, 1971/78).

But: computationally unfeasible for large-scale systems.

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Balanced Truncation

The Hankel Singular Values are Singular Values!

Theorem

Let P, Q be the controllability and observability Gramians of an LTI system Σ. Then the Hankel singular values $\Lambda(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the singular values of the Hankel operator associated to Σ.

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Proof: Hankel operator

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Theorem

Let P, Q be the controllability and observability Gramians of an LTI system Σ. Then the Hankel singular values $\Lambda(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the singular values of the Hankel operator associated to Σ.

Proof: Singular values of $\mathcal{H}$ = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^*\mathcal{H}u_-(t) = B^T e^{-A^T t} Qz \doteq \sigma^2 u_-(t).$$

$\Longrightarrow u_-(t) = \frac{1}{\sigma^2} B^T e^{-A^T t} Qz \Longrightarrow$ (recalling $z = \int_{-\infty}^{0} e^{-A\tau} B u_-(\tau)\, d\tau$)

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Balanced Truncation

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$\Longleftrightarrow \quad PQz = \sigma^2 z.$ $\square$

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Theorem

Let the reduced-order system $\hat{\Sigma} : (\hat{A}, \hat{B}, \hat{C}, \hat{D})$ with $r \leq \hat{n}$ be computed by balanced truncation. Then the reduced-order model $\hat{\Sigma}$ is balanced, stable, minimal, and its HSVs are $\sigma_1, \ldots, \sigma_r$.

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Proof: Note that in balanced coordinates, the Gramians are diagonal and equal to

$$\mathrm{diag}(\Sigma_1, \Sigma_2) = \mathrm{diag}(\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_n).$$

Hence, the Gramian satisfies

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} + \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}^T = 0,$$

whence we obtain the "controllability Lyapunov equation" of the reduced-order system,

$$A_{11}\Sigma_1 + \Sigma_1 A_{11}^T + B_1 B_1^T = 0.$$

The result follows from $\hat{A} = A_{11}$, $\hat{B} = B_1$, $\Sigma_1 > 0$, the solution theory of Lyapunov equations and the analogous considerations for the observability Gramian. (Minimality is a simple consequence of $\hat{P} = \Sigma_1 = \hat{Q} > 0$.)

Singular Perturbation Approximation (aka Balanced Residualization)

Assume the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \quad y = [\, C_1, \, C_2 \,] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du$$

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Particularly, if $G(0) = \hat{G}(0)$ ("zero steady-state error") is required, one can apply the same condensation technique as in Guyan reduction: instead of $x_2 = 0$, set $\dot{x}_2 = 0$. This yields the reduced-order model

$$\begin{aligned} \dot{x}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u, \\ y &= (C_1 - C_2A_{22}^{-1}A_{21})x_1 + (D - C_2A_{22}^{-1}B_2)u, \end{aligned}$$

with

- the same properties as the reduced-order model w.r.t. stability, minimality, error bound, but $\hat{D} \neq D$;
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Note:

- A_{22} invertible as in balanced coordinates, $A_{22}\Sigma_2 + \Sigma_2 A_{22}^T + B_2 B_2^T = 0$ and (A_{22}, B_2) controllable, $\Sigma_2 > 0 \;\Rightarrow\; A_{22}$ stable.
- If the original system is not balanced, first compute a minimal realization by applying balanced truncation with $r = \hat{n}$.

Balancing-Related Methods

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \dots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \dots \geq \sigma_n > 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

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Classical Balanced Truncation (BT) [Mullis/Roberts '76, Moore '81]

- P = controllability Gramian of system given by (A, B, C, D).
- Q = observability Gramian of system given by (A, B, C, D).
- P, Q solve dual Lyapunov equations

$$AP + PA^T + BB^T \;=\; 0, \qquad A^T Q + QA + C^T C \;=\; 0.$$

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LQG Balanced Truncation (LQGBT) [Jonckheere/Silverman '83]

- $P/Q =$ controllability/observability Gramian of closed-loop system based on LQG compensator.
- P, Q solve dual algebraic Riccati equations (AREs)

$$\begin{aligned} 0 &= AP + PA^T - PC^T CP + B^T B, \\ 0 &= A^T Q + QA - QBB^T Q + C^T C. \end{aligned}$$

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Balanced Stochastic Truncation (BST) [Desai/Pal '84, Green '88]

- $P =$ controllability Gramian of system given by (A, B, C, D), i.e., solution of Lyapunov equation $AP + PA^T + BB^T = 0$.
- $Q =$ observability Gramian of right spectral factor of power spectrum of system given by (A, B, C, D), i.e., solution of ARE

$$\hat{A}^T Q + Q\hat{A} + QB_W(DD^T)^{-1}B_W^T Q + C^T(DD^T)^{-1}C = 0,$$

where $\hat{A} := A - B_W(DD^T)^{-1}C$, $B_W := BD^T + PC^T$.

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Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

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Positive-Real Balanced Truncation (PRBT) [Green '88]

- Based on positive-real equations, related to positive real (Kalman-Yakubovich-Popov-Anderson) lemma.
- P, Q solve dual AREs

$$\begin{aligned} 0 &= \bar{A}P + P\bar{A}^T + PC^T\bar{R}^{-1}CP + B\bar{R}^{-1}B^T, \\ 0 &= \bar{A}^T Q + Q\bar{A} + QB\bar{R}^{-1}B^T Q + C^T\bar{R}^{-1}C, \end{aligned}$$

where $\bar{R} = D + D^T$, $\bar{A} = A - B\bar{R}^{-1}C$.

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Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

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Other Balancing-Based Methods

- Bounded-real balanced truncation (BRBT) – based on bounded real lemma [Opdenacker/Jonckheere '88];
- H_∞ balanced truncation (HinfBT) – closed-loop balancing based on H_∞ compensator [Mustafa/Glover '91].

Both approaches require solution of dual AREs.

- Frequency-weighted versions of the above approaches.

Balancing-Related Methods

Properties

- Guaranteed preservation of physical properties like
  - stability (all),
  - passivity (PRBT),
  - minimum phase (BST).
- Computable error bounds, e.g.,

$$\text{BT:} \quad \|G - G_r\|_\infty \quad \leq 2 \sum_{j=r+1}^{n} \sigma_j^{BT},$$

$$\text{LQGBT:} \quad \|G - G_r\|_\infty \quad \leq \ 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}}$$

$$\text{BST:} \quad \|G - G_r\|_\infty \quad \leq \left(\prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \|G\|_\infty,$$

- Can be combined with singular perturbation approximation for steady-state performance.
- Computations can be modularized.

Outline

Solving Large-Scale Matrix Equations

Large-Scale Algebraic Lyapunov and Riccati Equations

Algebraic Riccati equation (ARE) for $A, G = G^T, W = W^T \in \mathbb{R}^{n \times n}$ given and $X \in \mathbb{R}^{n \times n}$ unknown:

$$0 = \mathcal{R}(X) := A^T X + XA - XGX + W.$$

$G = 0 \Longrightarrow$ Lyapunov equation:

$$0 = \mathcal{L}(X) := A^T X + XA + W.$$

Typical situation in model reduction and optimal control problems for semi-discretized PDEs:

- $n = 10^3 - 10^6$ ($\Longrightarrow 10^6 - 10^{12}$ unknowns!),
- A has sparse representation ($A = -M^{-1}S$ for FEM),
- G, W low-rank with $G, W \in \{BB^T, C^T C\}$, where $B \in \mathbb{R}^{n \times m}$, $m \ll n$, $C \in \mathbb{R}^{p \times n}$, $p \ll n$.
- Standard (eigenproblem-based) $\mathcal{O}(n^3)$ methods are not applicable!

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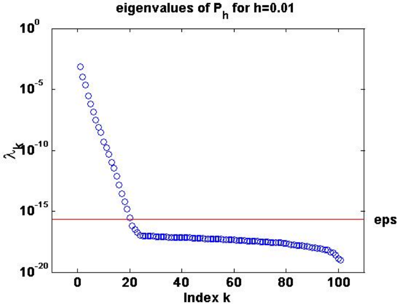
Solving Large-Scale Matrix Equations
Low-Rank Approximation

Consider spectrum of ARE solution (analogous for Lyapunov equations).

Example:

- Linear 1D heat equation with point control,
- $\Omega = [0, 1]$,
- FEM discretization using linear B-splines,
- $h = 1/100 \implies n = 101$.



Idea: $X = X^T \geq 0 \implies$

$$X = ZZ^T = \sum_{k=1}^{n} \lambda_k z_k z_k^T \approx Z^{(r)}(Z^{(r)})^T = \sum_{k=1}^{r} \lambda_k z_k z_k^T .$$

$\implies$ Goal: compute $Z^{(r)} \in \mathbb{R}^{n \times r}$ directly w/o ever forming X!

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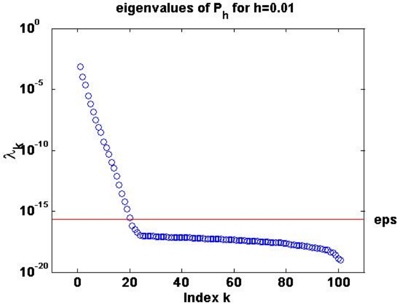

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Solving Large-Scale Matrix Equations

Linear Matrix Equations

Equations without symmetry

Sylvester equation

$$AX + XB = W$$

discrete Sylvester equation

$$AXB - X = W$$

with data $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times m}$, $W \in \mathbb{R}^{n \times m}$ and unknown $X \in \mathbb{R}^{n \times m}$.

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Stein equation (discrete Lyapunov equation)

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Here: focus on (Sylvester and) Lyapunov equations; analogous results and methods for discrete versions exist.

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Linear Matrix Equations

Solvability

Using the Kronecker (tensor) product, $AX + XB = W$ is equivalent to

$$\left((I_m \otimes A) + \left(B^T \otimes I_n\right)\right) \operatorname{vec}(X) = \operatorname{vec}(W).$$

Hence,

$$\text{Sylvester equation has a unique solution}$$

$$\Longleftrightarrow$$

$$M := (I_m \otimes A) + \left(B^T \otimes I_n\right) \text{ is invertible.}$$

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$$0 \notin \Lambda(M) = \Lambda\left((I_m \otimes A) + (B^T \otimes I_n)\right) = \{\lambda_j + \mu_k \mid \lambda_j \in \Lambda(A),\ \mu_k \in \Lambda(B)\}.$$

$$\Longleftrightarrow$$

$$\boxed{\Lambda(A) \cap \Lambda(-B) = \emptyset}$$

Corollary

A, B Hurwitz $\Longrightarrow$ Sylvester equation has unique solution.

Linear Matrix Equations

Solvability

Using the Kronecker (tensor) product, $AX + XB = W$ is equivalent to

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Linear Matrix Equations

Complexity Issues

Solving the Sylvester equation

$$AX + XB = W$$

via the equivalent linear system of equations

$$\left((I_m \otimes A) + \left(B^T \otimes I_n\right)\right) \operatorname{vec}(X) = \operatorname{vec}(W)$$

requires

- LU factorization of $nm \times nm$ matrix; for $n \approx m$, complexity is $\frac{2}{3}n^6$;
- storing $n \cdot m$ unknowns: for $n \approx m$ we have n^2 data for X, but up to n^4 data for triangular factors!

Example

$n = m = 1,000 \Rightarrow$ Gaussian elimination on an Intel core i7 (Westmere, 6 cores, 3.46 GHz $\rightsquigarrow$ 83.2 GFLOP peak) would take $>$ 94 DAYS and 7.3 TB of memory!

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Numerical Methods for Solving Lyapunov Equations
Traditional Methods

Bartels-Stewart method for Sylvester and Lyapunov equation (`lyap`);
Hessenberg-Schur method for Sylvester equations (`lyap`);
Hammarling's method for Lyapunov equations $AX + XA^T + GG^T = 0$ with A Hurwitz (`lyapchol`).

All based on the fact that if A, B^T are in Schur form, then

$$M = (I_m \otimes A) + \left(B^T \otimes I_n\right)$$

is block-upper triangular. Hence, solve $Mx = b$ by back-substitution.

- Clever implementation of back-substitution process requires $nm(n+m)$ flops.
- For Sylvester equations, B in Hessenberg form is enough ($\leadsto$ Hessenberg-Schur method).
- Hammarling's method computes Cholesky factor Y of X directly.
- All methods require Schur decomposition of A and Schur or Hessenberg decomposition of B $\Rightarrow$ need QR algorithm which requires $25n^3$ flops for Schur decomposition.

Not feasible for large-scale problems ($n > 10,000$).

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Numerical Methods for Solving Lyapunov Equations
The Sign Function Method

Definition

For $Z \in \mathbb{R}^{n\times n}$ with $\Lambda(Z) \cap \imath\mathbb{R} = \emptyset$ and Jordan canonical form

$$Z = S \begin{bmatrix} J^+ & 0 \\ 0 & J^- \end{bmatrix} S^{-1}$$

the matrix sign function is

$$\operatorname{sign}(Z) := S \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} S^{-1}.$$

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Lemma

Let $T \in \mathbb{R}^{n\times n}$ be nonsingular and Z as before, then

$$\mathrm{sign}\left(TZT^{-1}\right) = T\,\mathrm{sign}(Z)\,T^{-1}$$

Numerical Methods for Solving Lyapunov Equations
The Sign Function Method

Computation of $\operatorname{sign}(Z)$

$\operatorname{sign}(Z)$ is root of $I_n \Longrightarrow$ use Newton's method to compute it:

$$Z_0 \leftarrow Z, \qquad Z_{j+1} \leftarrow \frac{1}{2}\left(c_j Z_j + \frac{1}{c_j} Z_j^{-1}\right), \qquad j = 1, 2, \ldots$$

$\Longrightarrow \quad \operatorname{sign}(Z) = \lim_{j\to\infty} Z_j$.

$c_j > 0$ is scaling parameter for convergence acceleration and rounding error minimization, e.g.

$$c_j = \sqrt{\frac{\|Z_j^{-1}\|_F}{\|Z_j\|_F}},$$

based on "equilibrating" the norms of the two summands [Higham '86].

Solving Lyapunov Equations with the Matrix Sign Function Method

Key observation:
If $X \in \mathbb{R}^{n\times n}$ is a solution of $AX + XA^T + W = 0$, then

$$\underbrace{\begin{bmatrix} I_n & -X \\ 0 & I_n \end{bmatrix}}_{=T^{-1}} \underbrace{\begin{bmatrix} A & W \\ 0 & -A^T \end{bmatrix}}_{=:H} \underbrace{\begin{bmatrix} I_n & X \\ 0 & I_n \end{bmatrix}}_{=:T} = \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}.$$

Hence, if A is Hurwitz (i.e., asymptotically stable), then

$$\begin{aligned} \operatorname{sign}(H) &= \operatorname{sign}\left(T \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} T^{-1} \right) = T \operatorname{sign}\left(\begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} \right) T^{-1} \\ &= \begin{bmatrix} -I_n & 2X \\ 0 & I_n \end{bmatrix}. \end{aligned}$$

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Solving Lyapunov Equations with the Matrix Sign Function Method

Apply sign function iteration $Z \leftarrow \frac{1}{2}(Z + Z^{-1})$ to $H = \begin{bmatrix} A & W \\ 0 & -A^T \end{bmatrix}$:

$$H + H^{-1} = \begin{bmatrix} A & W \\ 0 & -A^T \end{bmatrix} + \begin{bmatrix} A^{-1} & A^{-1}WA^{-T} \\ 0 & -A^{-T} \end{bmatrix}$$

$\Longrightarrow$ Sign function iteration for Lyapunov equation:

$$\begin{aligned} A_0 &\leftarrow A, \quad A_{j+1} \leftarrow \tfrac{1}{2}\left(A_j + A_j^{-1}\right), \\ W_0 &\leftarrow G, \quad W_{j+1} \leftarrow \tfrac{1}{2}\left(W_j + A_j^{-1} W_j A_j^{-T}\right), \end{aligned} \qquad j = 0, 1, 2, \ldots.$$

Define $A_\infty := \lim_{j\to\infty} A_j$, $W_\infty := \lim_{j\to\infty} W_j$.

Theorem

If A is Hurwitz, then

$$A_\infty = -I_n \qquad \text{and} \qquad X = \frac{1}{2} W_\infty.$$

Solving Lyapunov Equations with the Matrix Sign Function Method

Factored form

Recall sign function iteration for $AX + XA^T + W = 0$:

$$\begin{array}{ll} A_0 \leftarrow A, & A_{j+1} \leftarrow \frac{1}{2}\left(A_j + A_j^{-1}\right), \\ W_0 \leftarrow G, & W_{j+1} \leftarrow \frac{1}{2}\left(W_j + A_j^{-1} W_j A_j^{-T}\right), \end{array} \qquad j = 0, 1, 2, \ldots.$$

Now consider the second iteration for $W = BB^T$, starting with $W_0 = BB^T =: B_0 B_0^T$:

$$\begin{aligned} \frac{1}{2}\left(W_j + A_j^{-1} W_j A_j^{-T}\right) &= \frac{1}{2}\left(B_j B_j^T + A_j^{-1} B_j B_j^T A_j^{-T}\right) \\ &= \frac{1}{2}\left[B_j \quad A_j^{-1} B_j\right]\left[B_j \quad A_j^{-1} B_j\right]^T. \end{aligned}$$

Hence, obtain factored iteration

$$B_{j+1} \leftarrow \frac{1}{\sqrt{2}}\left[B_j \quad A_j^{-1} B_j\right]$$

with $S := \frac{1}{\sqrt{2}} \lim_{j\to\infty} B_j$ and $X = SS^T$.

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Solving Lyapunov Equations with the Matrix Sign Function Method

Factored form **[B./Quintana-Ortí '97]**

Factored sign function iteration for $A(SS^T) + (SS^T)A^T + BB^T = 0$

$$\begin{array}{ll} A_0 \leftarrow A, \quad A_{j+1} \leftarrow \frac{1}{2}\left(A_j + A_j^{-1}\right), & \\ B_0 \leftarrow B, \quad B_{j+1} \leftarrow \frac{1}{\sqrt{2}}\left[B_j \quad A_j^{-1}B_j\right], & j = 0, 1, 2, \ldots. \end{array}$$

Remarks:

- To get both Gramians, run in parallel

$$C_{j+1} \leftarrow \frac{1}{\sqrt{2}} \begin{bmatrix} C_j \\ C_j A_j^{-1} \end{bmatrix}.$$

- To avoid growth in numbers of columns of B_j (or rows of C_j): column compression by RRLQ or truncated SVD.
- Several options to incorporate scaling, e.g., scale "A"-iteration only.
- Simple stopping criterion: $\|A_j + I_n\|_F \leq tol$.

Numerical Methods for Solving Lyapunov Equations
The ADI Method

Recall Peaceman Rachford ADI:
Consider $Au = s$ where $A \in \mathbb{R}^{n\times n}$ spd, $s \in \mathbb{R}^n$. ADI Iteration Idea:
Decompose $A = H + V$ with $H, V \in \mathbb{R}^{n\times n}$ such that

$$\begin{aligned}(H + pI)v &= r\\ (V + pI)w &= t\end{aligned}$$

can be solved easily/efficiently.

ADI Iteration

If H, V spd $\Rightarrow \exists p_k,\ k = 1, 2, \ldots$ such that

$$\begin{aligned} u_0 &= 0\\ (H + p_k I)u_{k-\frac{1}{2}} &= (p_k I - V)u_{k-1} + s\\ (V + p_k I)u_k &= (p_k I - H)u_{k-\frac{1}{2}} + s\end{aligned}$$

converges to $u \in \mathbb{R}^n$ solving $Au = s$.

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Numerical Methods for Solving Lyapunov Equations

The Lyapunov operator

$$\mathcal{L}: \quad X \quad \mapsto \quad AX + XA^T$$

can be decomposed into the linear operators

$$\mathcal{L}_H : X \mapsto AX, \qquad \mathcal{L}_V : X \mapsto XA^T.$$

In analogy to the standard ADI method we find the

ADI iteration for the Lyapunov equation [Wachspress '88]

$$\begin{aligned} X_0 &= 0 \\ (A + p_k I)X_{k-\frac{1}{2}} &= -W - X_{k-1}(A^T - p_k I) \\ (A + p_k I)X_k^T &= -W - X_{k-\frac{1}{2}}^T(A^T - p_k I). \end{aligned}$$

Numerical Methods for Solving Lyapunov Equations
Low-Rank ADI

Consider $AX + XA^T = -BB^T$ for stable A; $B \in \mathbb{R}^{n\times m}$ with $m \ll n$.

ADI iteration for the Lyapunov equation [Wachspress '95]

For $k = 1, \ldots, k_{\max}$

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Rewrite as one step iteration and factorize $X_k = Z_k Z_k^T$, $k = 0, \ldots, k_{\max}$

$$\begin{aligned} Z_0 Z_0^T &= 0 \\ Z_k Z_k^T &= -2p_k(A + p_k I)^{-1}BB^T(A + p_k I)^{-T} \\ &\quad +(A + p_k I)^{-1}(A - p_k I)Z_{k-1}Z_{k-1}^T(A - p_k I)^T(A + p_k I)^{-T} \end{aligned}$$

$\ldots \rightsquigarrow$ low-rank Cholesky factor ADI

[Penzl '97/'00, Li/White '99/'02, B./Li/Penzl '99/'08, Gugercin/Sorensen/Antoulas '03]

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Solving Large-Scale Matrix Equations
Numerical Methods for Solving Lyapunov Equations

$$Z_k = [\sqrt{-2p_k}(A + p_k I)^{-1}B,\ (A + p_k I)^{-1}(A - p_k I)Z_{k-1}]$$

[Penzl '00]

Observing that $(A - p_i I)$, $(A + p_k I)^{-1}$ commute, we rewrite $Z_{k_{\max}}$ as

$$Z_{k_{\max}} = [z_{k_{\max}},\ P_{k_{\max}-1}z_{k_{\max}},\ P_{k_{\max}-2}(P_{k_{\max}-1}z_{k_{\max}}),\ \ldots,\ P_1(P_2 \cdots P_{k_{\max}-1}z_{k_{\max}})],$$

[Li/White '02]

where

$$z_{k_{\max}} = \sqrt{-2p_{k_{\max}}}(A + p_{k_{\max}} I)^{-1}B$$

and

$$P_i := \frac{\sqrt{-2p_i}}{\sqrt{-2p_{i+1}}}\left[I - (p_i + p_{i+1})(A + p_i I)^{-1}\right].$$

Solving Large-Scale Matrix Equations
Numerical Methods for Solving Lyapunov Equations

$$Z_k = [\sqrt{-2p_k}(A + p_k I)^{-1}B,\ (A + p_k I)^{-1}(A - p_k I)Z_{k-1}]$$

[Penzl '00]

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Numerical Methods for Solving Lyapunov Equations

Lyapunov equation $0 = AX + XA^T + BB^T$.

Algorithm [Penzl '97/'00, Li/White '99/'02, B. 04, B./Li/Penzl '99/'08]

$$V_1 \leftarrow \sqrt{-2\,\mathrm{re}\,p_1}(A + p_1 I)^{-1}B, \qquad Z_1 \leftarrow V_1$$

FOR $k = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{re}\,p_k}{\mathrm{re}\,p_{k-1}}}\left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1}V_{k-1}\right)$$

$$Z_k \leftarrow \left[\begin{array}{cc} Z_{k-1} & V_k \end{array}\right]$$

$$Z_k \leftarrow \mathrm{rrlq}(Z_k, \tau) \qquad \text{column compression}$$

At convergence, $Z_{k_{\max}} Z_{k_{\max}}^T \approx X$, where (without column compression)

$$Z_{k_{\max}} = \left[\begin{array}{ccc} V_1 & \ldots & V_{k_{\max}} \end{array}\right], \quad V_k = \boxed{} \in \mathbb{C}^{n \times m}.$$

Note: Implementation in real arithmetic possible by combining two steps [B./Li/Penzl '99/'08] or using new idea employing the relation of 2 consecutive complex factors [B./Kürschner/Saak '11].

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Numerical Results for ADI

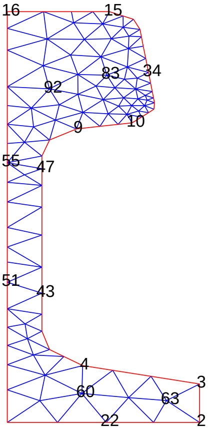
Optimal Cooling of Steel Profiles

- Mathematical model: boundary control for linearized 2D heat equation.

$$
\begin{aligned}
c \cdot \rho \frac{\partial}{\partial t} x &= \lambda \Delta x, && \xi \in \Omega \\
\lambda \frac{\partial}{\partial n} x &= \kappa(u_k - x), && \xi \in \Gamma_k,\ 1 \leq k \leq 7, \\
\frac{\partial}{\partial n} x &= 0, && \xi \in \Gamma_7.
\end{aligned}
$$

$\Longrightarrow m = 7, q = 6.$

- FEM Discretization, different models for initial mesh ($n = 371$), 1, 2, 3, 4 steps of mesh refinement $\Rightarrow$ $n = 1357, 5177, 20209, 79841.$



Source: Physical model: courtesy of Mannesmann/Demag.

Math. model: Tröltzsch/Unger 1999/2001, Penzl 1999, Saak 2003.

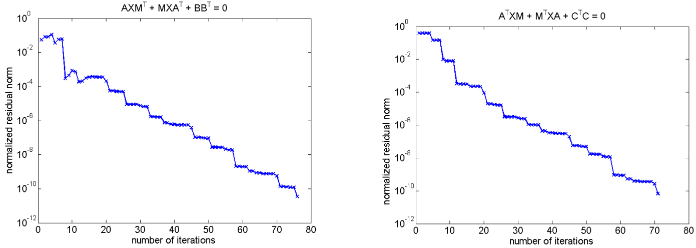
Numerical Results for ADI

Optimal Cooling of Steel Profiles

- Solve dual Lyapunov equations needed for balanced truncation, i.e.,

$$APM^T + MPA^T + BB^T = 0, \quad A^TQM + M^TQA + C^TC = 0,$$

  for $n = 79,841$.
- 25 shifts chosen by Penzl heuristic from 50/25 Ritz values of A of largest/smallest magnitude, no column compression performed.
- No factorization of mass matrix required.
- Computations done on Core2Duo at 2.8GHz with 3GB RAM and 32Bit-MATLAB.



CPU times: 626 / 356 sec.

Numerical Results for ADI

Scaling **Computations by Martin Köhler '10**

- $A \in \mathbb{R}^{n \times n} \equiv$ FDM matrix for 2D heat equation on $[0, 1]^2$ (LYAPACK benchmark `demo_l1`, $m = 1$).
- 16 shifts chosen by Penzl heuristic from 50/25 Ritz values of A of largest/smallest magnitude.
- Computations on 2 dual core Intel Xeon 5160 with 16 GB RAM using M.E.S.S. (http://svncsc.mpi-magdeburg.mpg.de/trac/messtrac/).

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CPU Times

n	M.E.S.S.[1] (C)	LyaPack	M.E.S.S. (MATLAB)
100	0.023	0.124	0.158
625	0.042	0.104	0.227
2,500	0.159	0.702	0.989
10,000	0.965	6.22	5.644
40,000	11.09	71.48	34.55
90,000	34.67	418.5	90.49
160,000	109.3	out of memory	219.9
250,000	193.7	out of memory	403.8
562,500	930.1	out of memory	1216.7
1,000,000	2220.0	out of memory	2428.6

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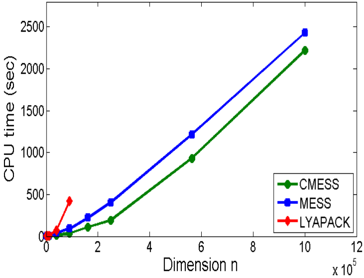

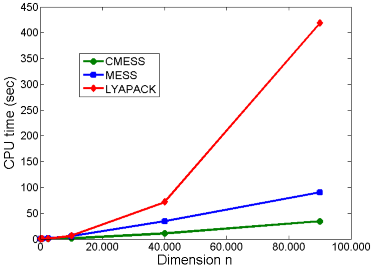

Note: for $n = 1,000,000$, first sparse LU needs $\sim 1,100$ sec., using UMFPACK this reduces to 30 sec.

Factored Galerkin-ADI Iteration

Lyapunov equation $0 = AX + XA^T + BB^T$

Projection-based methods for Lyapunov equations with $A + A^T < 0$:

1. Compute orthonormal basis $\mathrm{range}(Z)$, $Z \in \mathbb{R}^{n \times r}$, for subspace $\mathcal{Z} \subset \mathbb{R}^n$, $\dim \mathcal{Z} = r$.
2. Set $\hat{A} := Z^T AZ$, $\hat{B} := Z^T B$.
3. Solve small-size Lyapunov equation $\hat{A}\hat{X} + \hat{X}\hat{A}^T + \hat{B}\hat{B}^T = 0$.
4. Use $X \approx Z\hat{X}Z^T$.

Examples:

- Krylov subspace methods, i.e., for $m = 1$:

$$\mathcal{Z} = \mathcal{K}(A, B, r) = \mathrm{span}\{B, AB, A^2B, \ldots, A^{r-1}B\}$$

[Saad '90, Jaimoukha/Kasenally '94, Jbilou '02–'08].

- K-PIK [Simoncini '07],

$$\mathcal{Z} = \mathcal{K}(A, B, r) \cup \mathcal{K}(A^{-1}, B, r).$$

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4. Use $X \approx Z\hat{X}Z^T$.

Examples:

- ADI subspace [B./R.-C. Li/Truhar '08]:

$$\mathcal{Z} = \mathrm{colspan}\left[\begin{array}{ccc} V_1, & \ldots, & V_r \end{array}\right].$$

Note:

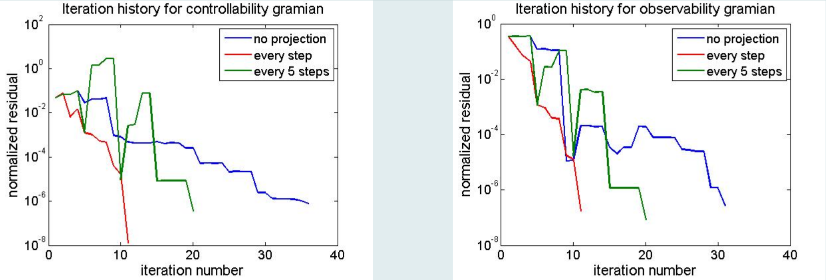
1. ADI subspace is rational Krylov subspace [J.-R. Li/White '02].
2. Similar approach: ADI-preconditioned global Arnoldi method [Jbilou '08].

Numerical Methods for Solving Lyapunov Equations
Numerical examples for Galerkin-ADI

FEM semi-discretized control problem for parabolic PDE:

- optimal cooling of rail profiles,
- $n = 20,209,\ m = 7,\ q = 6.$

Good ADI shifts



CPU times: 80s (projection every 5th ADI step) vs. 94s (no projection).

Computations by Jens Saak '10.

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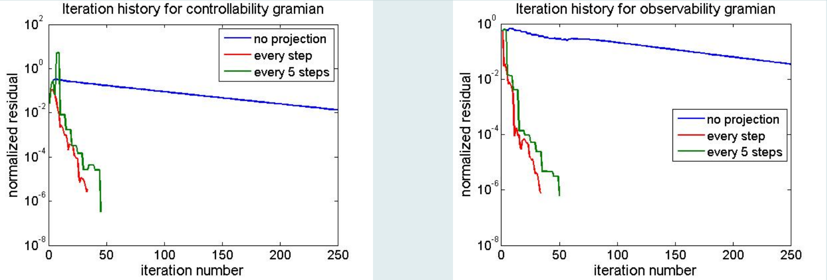


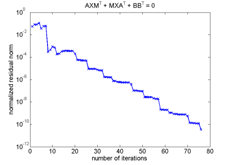
CPU times: 368s (projection every 5th ADI step) vs. 1207s (no projection).

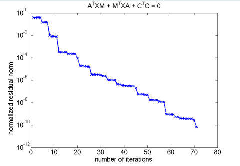
Computations by Jens Saak '10.

Numerical Methods for Solving Lyapunov Equations

Numerical examples for Galerkin-ADI: optimal cooling of rail profiles, $n = 79,841$.

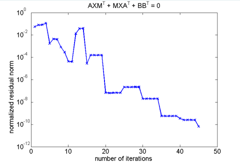
M.E.S.S. w/o Galerkin projection and column compression

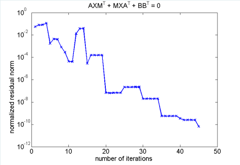




Rank of solution factors: 532 / 426

M.E.S.S. with Galerkin projection and column compression



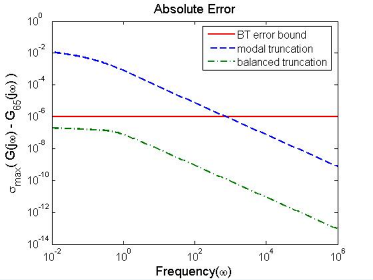


Rank of solution factors: 269 / 205

Solving Large-Scale Matrix Equations

Numerical example for BT: Optimal Cooling of Steel Profiles

$n = 1,357$, Absolute Error



– BT model computed with sign function method,

– MT w/o static condensation, same order as BT model.

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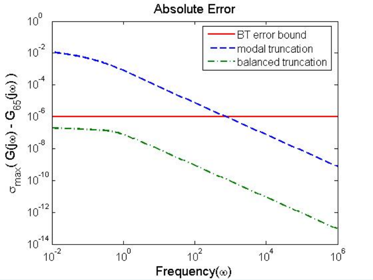

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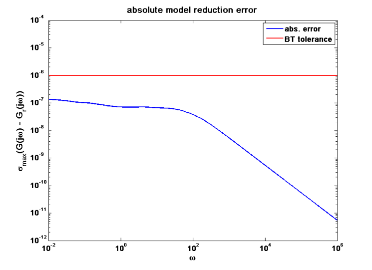
$n = 79,841$, Absolute Error

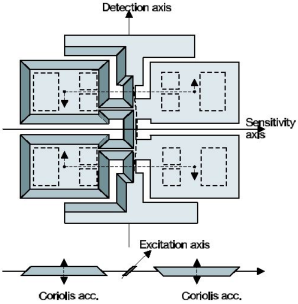


- BT model computed using M.E.S.S. in MATLAB,
- dualcore, computation time: <10 min.

Solving Large-Scale Matrix Equations

Numerical example for BT: Microgyroscope (Butterfly Gyro)

- Vibrating micro-mechanical gyroscope for inertial navigation.
- Rotational position sensor.



- By applying AC voltage to electrodes, wings are forced to vibrate in anti-phase in wafer plane.
- Coriolis forces induce motion of wings out of wafer plane yielding sensor data.

Source: The Oberwolfach Benchmark Collection `http://www.imtek.de/simulation/benchmark`

Courtesy of D. Billger (Imego Institute, Göteborg), Saab Bofors Dynamics AB.

Solving Large-Scale Matrix Equations

Numerical example for BT: Microgyroscope (Butterfly Gyro)

- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
  $\rightsquigarrow$ $n = 34,722$, $m = 1$, $q = 12$.
- Reduced model computed using SpaRed, $r = 30$.

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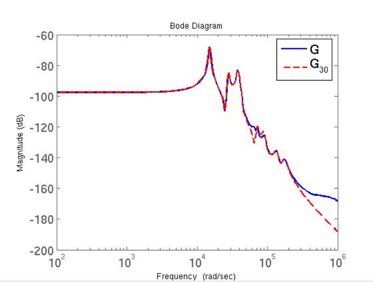
Frequency Repsonse Analysis



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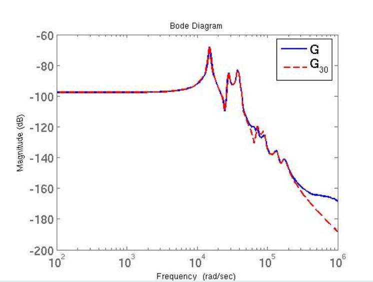
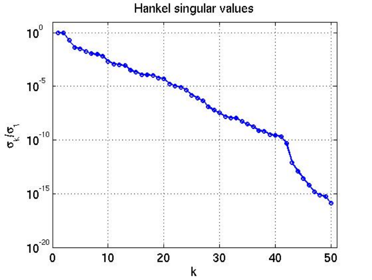


Hankel Singular Values



Solving Large-Scale Algebraic Riccati Equations

Theory **[Lancaster/Rodman '95]**

Theorem

Consider the (continuous-time) algebraic Riccati equation (ARE)

$$0 = \mathcal{R}(X) = C^T C + A^T X + XA - XBB^T X,$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$, (A, B) stabilizable, (A, C) detectable.

Then:

(a) There exists a unique stabilizing $X_* \in \{X \in \mathbb{R}^{n\times n} \mid \mathcal{R}(X) = 0\}$, i.e., $\Lambda(A - BB^T X_*) \in \mathbb{C}^-$.

(b) $X_* = X_*^T \geq 0$ and $X_* \geq X$ for all $X \in \{X \in \mathbb{R}^{n\times n} \mid \mathcal{R}(X) = 0\}$.

(c) If (A, C) observable, then $X_* > 0$.

(d) $\operatorname{span}\left\{\begin{bmatrix} I_n \\ -X_* \end{bmatrix}\right\}$ is the unique invariant subspace of the Hamiltonian matrix

$$H = \begin{bmatrix} A & BB^T \\ C^T C & -A^T \end{bmatrix}$$

corresponding to $\Lambda(H) \cap \mathbb{C}^-$.

Solving Large-Scale Algebraic Riccati Equations

Numerical Methods [Bini/Iannazzo/Meini '12]

Numerical Methods (incomplete list)

- Invariant subspace methods ($\rightsquigarrow$ eigenproblem for Hamiltonian matrix):
  - Schur vector method (`care`) [Laub '79]
  - Hamiltonian SR algorithm [Bunse-Gerstner/Mehrmann '86]
  - Symplectic URV-based method [B./Mehrmann/Xu '97/'98, Chu/Liu/Mehrmann '07]
- Spectral projection methods
  - Sign function method [Roberts '71, Byers '87]
  - Disk function method [Bai/Demmel/Gu '94, B. '97]
- (rational, global) Krylov subspace techniques [Jaimoukha/Kasenally '94, Jbilou '03/'06, Heyouni/Jbilou '09]
- Newton's method
  - Kleinman iteration [Kleinman '68]
  - Line search acceleration [B./Byers '98]
  - Newton-ADI [B./J.-R. Li/Penzl '99/'08]
  - Inexact Newton [Feitzinger/Hylla/Sachs '09,B./Heinkenschloss/Saak/Weichelt '15]

Solving Large-Scale Matrix Equations
Software

Lyapack [Penzl 2000]

MATLAB toolbox for solving

– Lyapunov equations and algebraic Riccati equations,
– model reduction and LQR problems.

Main work horse: Low-rank ADI and Newton-ADI iterations.

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- Includes solvers for large-scale differential Riccati equations (based on Rosenbrock and BDF methods).
- Many algorithmic improvements:
  – new ADI parameter selection,
  – column compression based on RRQR,
  – more efficient use of direct solvers,
  – treatment of generalized systems without factorization of the mass matrix,
  – new ADI versions avoiding complex arithmetic etc.
- C and MATLAB versions.

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Topics Not Covered

- Special methods for second-order (mechanical) systems.
- Extensions to bilinear and stochastic systems.
- Balanced truncation for discrete-time systems.
- Extensions to descriptor systems $E\dot{x} = Ax + Bu$, E singular.
- Frequency-limited/-weighted balanced truncation.
- Application to parametric model reduction:

$$\dot{x} = A(p)x + B(p)u, \quad y = C(p)x,$$

  where $p \in \mathbb{R}^d$ is a free parameter vector; parameters should be preserved in the reduced-order model.

Further Reading — Balanced Truncation

1. B. C. Moore.
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