Model Order Reduction via System Balancing

Exercise 1 (Controllability of dynamical systems)
Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Show that the following statements are equivalent:

a) the pair (A, B) is controllable
(i.e. for all times $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$, and states $x_0, x_1 \in \mathbb{R}^n$, there exists $u(t)$ such that the solution of the initial value problem $\dot{x}(t) = Ax(t) + Bu(t)$, $x(t_0) = x_0$, satisfies $x(t_1) = x_1$),

b) the controllability matrix $\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n\times nm}$ has full rank n,

c) the controllability Gramian

$$P(t) = \int_0^t e^{A\tau} BB^T e^{A^T\tau} \, \mathrm{d}\tau$$

is positive definite for all $t > 0$.

Exercise 2 (The (infinite) controllability Gramian and a Lyapunov equation)
Let $A \in \mathbb{R}^{n\times n}$ be stable and $Q \in \mathbb{R}^{n\times n}$. Prove that

$$X = \int_0^\infty e^{At} Q e^{A^T t} \, \mathrm{d}t$$

is the unique solution of the Lyapunov equation

$$AX + XA^T + Q = 0.$$

Exercise 3 (Stability, controllability, and the Lyapunov equation)
Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Prove that, of the following three statements, any two imply the third:

a) A is a stable matrix,

b) the pair (A, B) is controllable,

c) the Lyapunov equation $AP + PA^T + BB^T = 0$ has a positive definite solution P.

Exercise 4 (Properties of the matrix sign function)
For $Z \in \mathbb{C}^{n\times n}$ with no eigenvalues on the imaginary axis and a Jordan canonical form

$$Z = S \begin{bmatrix} J^+ & \\ & J^- \end{bmatrix} S^{-1},$$

where $J^+ \in \mathbb{C}^{k\times k}$ and $J^- \in \mathbb{C}^{(n-k)\times(n-k)}$ respectively have eigenvalues in $\mathbb{C}_+$ and $\mathbb{C}_-$, we define the matrix sign function as

$$Z = S \begin{bmatrix} I_k & \\ & -I_{n-k} \end{bmatrix} S^{-1}.$$

Show that:

a) the matrix sign function is well-defined,

b) $\operatorname{sign}(T^{-1}ZT) = T^{-1} \operatorname{sign}(Z)T$ for all nonsingular $T \in \mathbb{C}^{n\times n}$,

c) if Z is stable, then $\mathrm{sign}(Z) = -I_n$ and $\mathrm{sign}(-Z) = I_n$,

d) $\mathrm{sign}(Z)^2 = I_n$, i.e. $\mathrm{sign}(Z)$ is a square root of the identity matrix,

e) the Newton iteration $Z_0 = Z$, $Z_{i+1} = \frac{1}{2}(Z_i + Z_i^{-1})$, $i = 0, 1, 2, \ldots$, is a Newton iteration applied to the function $F(X) = X^2 - I$.

Exercise 5 (Solving Sylvester equations via the matrix sign function)

Consider the Sylvester equation

$$AX + XB + C = 0, \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, and $C \in \mathbb{R}^{n\times m}$. Assume that A and B are stable matrices and that X is the solution of the equation (1).

a) Show that

$$\mathrm{sign}\left(\begin{bmatrix} A & C \\ 0 & -B \end{bmatrix}\right) = \begin{bmatrix} -I_n & 2X \\ 0 & I_m \end{bmatrix}.$$

Hint: Compute $T^{-1}\begin{bmatrix} A & C \\ 0 & -B \end{bmatrix} T$, for $T = \begin{bmatrix} I_n & X \\ 0 & I_m \end{bmatrix}$.

b) Show that instead of iterating on $\begin{bmatrix} A & C \\ 0 & -B \end{bmatrix}$, one can compute X via an iteration on A, B, C.

Exercise 6 (Implementing a Lyapunov equation solver)

Our goal here is to implement a solver, using matrix sign function Newton iteration, for the Lyapunov equation

$$AX + XA^T + W = 0, \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ is a stable matrix.

a) Derive the iteration method

$$\begin{aligned} A_0 &= A, & A_{i+1} &= \frac{1}{2}\left(A_i + A_i^{-1}\right), \\ W_0 &= W, & W_{i+1} &= \frac{1}{2}\left(W_i + A_i^{-1} W_i A_i^{-T}\right), \end{aligned}$$

for the equation (2) using the solution of the Exercise 5 b). Show that $A_i \to -I_n$ and $W_i \to 2X$.

b) Implement a function `lyap_sgn`, applying the above iteration, with the matrices A and W, the maximum number of iterations `maxit`, and the tolerance `tol` for the stopping criterion $\|A_i + I_n\|_F <$ `tol` as inputs.

c) Test your implementation on random examples by computing the relative error

$$\frac{\|AX + XA^T + W\|_F}{\|W\|_F}$$

and plotting how $\|A_i + I_n\|_F$ varies across iterations. Check if the approximate solution you find is symmetric (e.g. by computing $\|X - X^T\|_F$) for symmetric W.

Exercise 7 (Model reduction by balanced truncation)

Here we apply the balanced truncation method to the Clamped Beam model from the NICONET benchmark collection (you need to download `beam.mat` from [1]).

a) Compute the controllability and observability Gramians by solving the Lyapunov equations

$$\begin{aligned} AP + PA^T + BB^T &= 0, \\ A^TQ + QA + C^TC &= 0, \end{aligned}$$

using the function `lyap_sgn` you implemented in the previous Exercise.

b) Compute factorizations $P = S^T S$ and $Q = R^T R$.

c) Compute the singular value decomposition $SR^T = U\Sigma V^T$.

d) Plot the Hankel singular values.

e) Find the reduced order model $(A_r, B_r, C_r) = (W_r^T A V_r, W_r^T B, C V_r)$, where

$$\begin{aligned} V_r &= S^T U(:, 1:r)\Sigma(1:r, 1:r)^{-\frac{1}{2}}, \\ W_r &= R^T V(:, 1:r)\Sigma(1:r, 1:r)^{-\frac{1}{2}}, \end{aligned}$$

for some r.

f) Draw the log-log plots of $\omega \mapsto |H(i\omega)|$ and $\omega \mapsto |H_r(i\omega)|$, where

$$\begin{aligned} H(s) &= C(sI_n - A)^{-1}B, \\ H_r(s) &= C_r(sI_r - A_r)^{-1}B_r, \end{aligned}$$

are the transfer functions of the original and reduced model. Use 1000 logarithmically distributed sample points over the frequency interval $\omega \in [10^{-2}, 10^4]$.

g) Draw the log-log plot of $\omega \mapsto |H(i\omega) - H_r(i\omega)|$, same as in f), with a horizontal line for the upper bound of the $\mathcal{H}_\infty$-error using Hankel singular values.

Exercise 8 (Balancing-free square root (BFSR) method)

For numerical reasons, the balancing-free square root (BFSR) algorithm is preferred to the method used in the previous Exercise. The difference is in the part e).

a) Compute the projection matrices

$$V_r = P_1 \text{ and } W_r = Q_1 \left(P_1^T Q_1\right)^{-1},$$

where

$$S^T U_1 = P_1 \widehat{R} \text{ and } R^T V_1 = Q_1 \widetilde{R},$$

with $P_1, Q_1 \in \mathbb{R}^{n\times r}$ orthogonal and $\widehat{R}, \widetilde{R} \in \mathbb{R}^{r\times r}$ upper-triangular.

b) Show that the reduced order system is equivalent to a balanced system and that it satisfies the same error bound as the one obtained by the standard square root balanced truncation method.

Exercise 9 (Low-rank Lyapunov equation solver)

It is possible to combine parts a) and b) in Exercise 7.

a) For the Lyapunov equation

$$AX + XA^T + BB^T = 0,$$

derive the iteration method

$$\begin{aligned} A_0 &= A, \quad A_{i+1} = \frac{1}{2}\left(A_i + A_i^{-1}\right), \\ B_0 &= B, \quad B_{i+1} = \frac{1}{\sqrt{2}}\begin{bmatrix} B_i & A_i^{-1}B_i \end{bmatrix}, \end{aligned}$$

by setting $W_i = B_i B_i^T$ in Exercise 6 a).

b) Since B_{i+1} has the twice the number of column as B_i, it is necessary to include column compression in the iterations. Implement a function `col_comp` that will perform this for an arbitrary matrix, using rank-revealing LQ decomposition or SVD, with specified error tolerance.

c) Implement a Lyapunov equation solver `lyap_sgn_fac`, using the above iterations with column compression.

Exercise 10 (Solving algebraic Riccati equations via the matrix sign function)
Motivated by balancing-related methods such as LQG balanced truncation, let us consider the algebraic Riccati equation

$$AX + XA^T - XFX + G = 0,$$

where $A \in \mathbb{R}^{n\times n}$ and $F = F^T,\ G = G^T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices and (A, F) is stabilizable. Let

$$M = \begin{bmatrix} A & G \\ F & -A^T \end{bmatrix}$$

and assume that the matrix sign function of M is partitioned as

$$\operatorname{sign}(M) = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}.$$

Show that

$$\begin{bmatrix} I_n - Z_{11} \\ Z_{21} \end{bmatrix} X = \begin{bmatrix} Z_{12} \\ I_n - Z_{22} \end{bmatrix}.$$

Hint: First show that

$$M = \begin{bmatrix} I_n - XQ & X \\ -Q & I_n \end{bmatrix} \begin{bmatrix} A - XF & 0 \\ 0 & -(A - XF)^T \end{bmatrix} \begin{bmatrix} I_n - XQ & X \\ -Q & I_n \end{bmatrix}^{-1},$$

where Q solves $(A - XF)^T Q + Q(A - XF) + F = 0$. Then make use of the properties of the matrix sign function.

References

[1] http://slicot.org/20-site/126-benchmark-examples-for-model-reduction