High dimensional approximation of parametric PDE's
Theory and Algorithms

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Overview

1. Introduction to the main themes.
2. Sparse polynomial approximation of parametric/stochastic PDEs.
3. Sparse polynomial algorithms.
4. Reduced basis methods (if time permits).

References

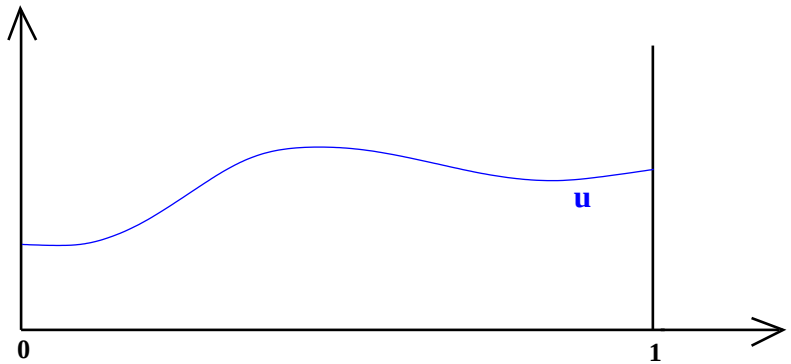
R. DeVore, "Nonlinear approximation", Acta Numerica, 1998.

A. Cohen, R. DeVore and C. Schwab, "Analytic regularity and polynomial approximation of parametric and stochastic PDEs", Analysis and Application, 2011.

A. Cohen and R. DeVore, "High dimensional approximation of parametric PDEs", Acta Numerica, 2015.

The curse of dimensionality

Consider a continuous function $y \mapsto u(y)$ with $y \in [0,1]$. Sample at equispaced points. Reconstruct, for example by piecewise linear interpolation.



Error in terms of point spacing $h > 0$: if u has C^2 smoothness

$$\|u - R(u)\|_{L^\infty} \leq C\|u''\|_{L^\infty} h^2.$$

Using piecewise polynomials of higher order, if u has C^m smoothness

$$\|u - R(u)\|_{L^\infty} \leq C\|u^{(m)}\|_{L^\infty} h^m.$$

In terms of the number of samples $n \sim h^{-1}$, the error is estimated by n^{-m}.

In d dimensions : $u(y) = u(y_1, \cdots, y_d)$ with $y \in [0,1]^d$. With a uniform sampling, we still have

$$\|u - R(u)\|_{L^\infty} \leq C\Big(\sup_{|\alpha|=m} \|\partial^\alpha u\|_{L^\infty}\Big) h^m,$$

but the number of samples is now $n \sim h^{-d}$, and the error estimate is in $n^{-m/d}$.

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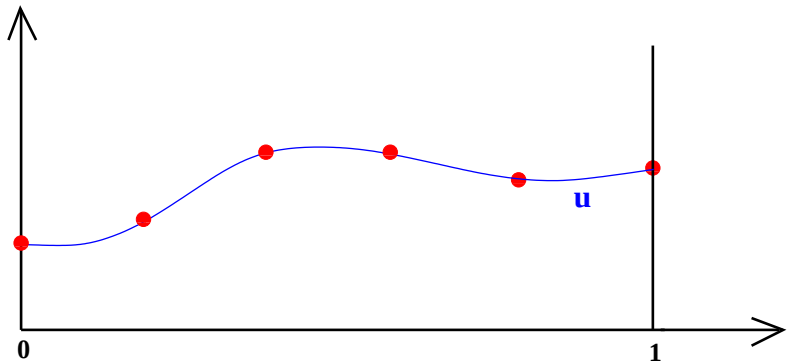

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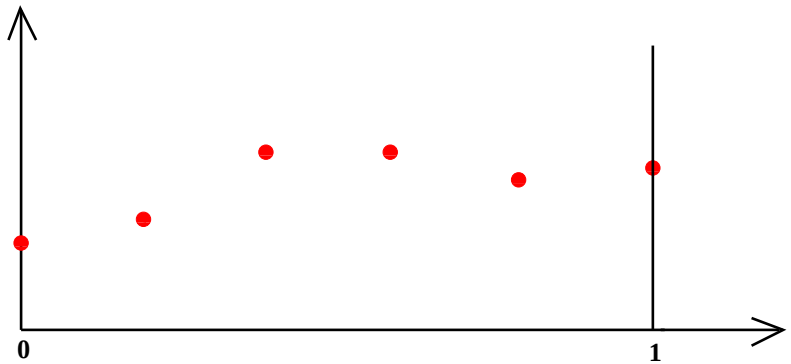

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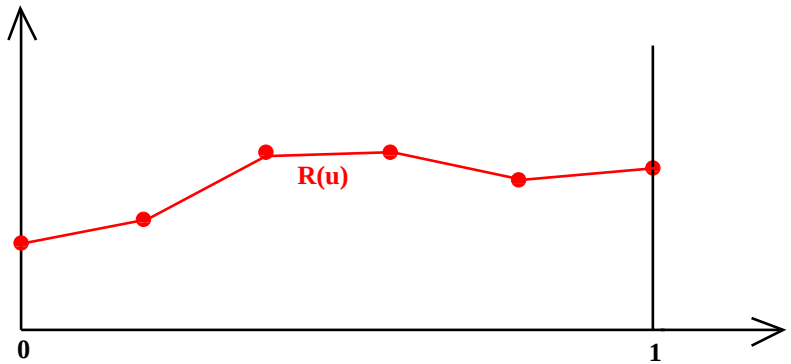

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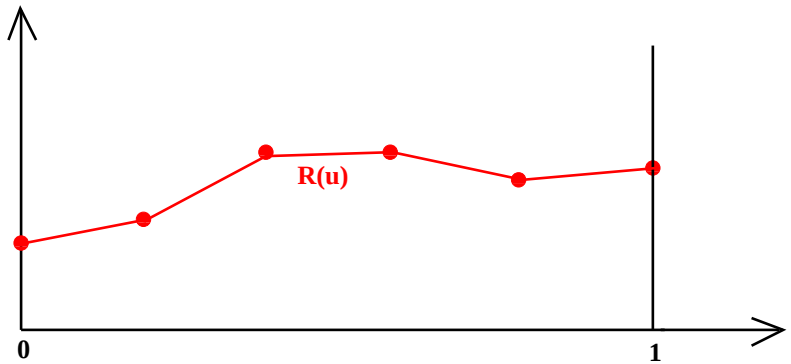

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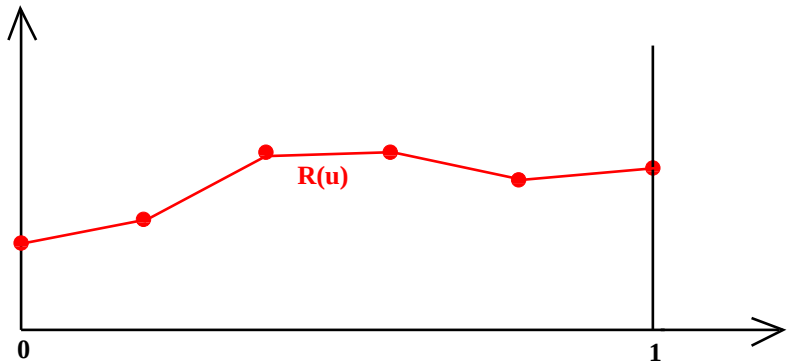

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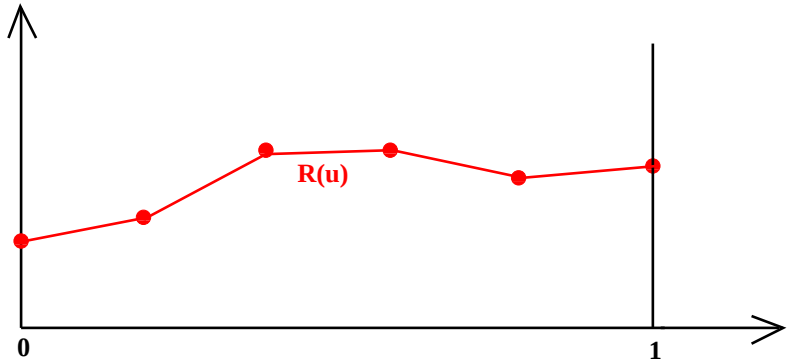

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Other sampling/reconstruction methods cannot do better

Can be explained by *n*-width

Let X be a normed space and $\mathcal{K} \subset X$ a compact set.

Linear n-width (Kolmogorov) :

$$d_N(\mathcal{K})_X := \inf_{\dim(E)=n} \max_{u \in \mathcal{K}} \min_{v \in E} \|u - v\|_X.$$

Benchmark for linear approximation methods applied to the elements from $\mathcal{K}$.

If $X = L^\infty([0,1]^d)$ and $\mathcal{K}$ is the unit ball of $C^m([0,1]^d)$ it is known that

$$cn^{-m/d} \leq d_n(\mathcal{K})_X \leq Cn^{-m/d}.$$

Upper bound : approximation by a specific method.

Lower bound : diversity in $\mathcal{K}$.

Exponential growth in d of the needed complexity to reach a given accuracy.

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Use a notion of nonlinear n-width (Alexandrov, DeVore-Howard-Micchelli).

Consider maps $E : \mathcal{K} \mapsto \mathbb{R}^n$ (encoding) and $R : \mathbb{R}^n \mapsto X$ (reconstruction).

Introducing the distorsion of the pair (E, R) over $\mathcal{K}$

$$\max_{u \in \mathcal{K}} \|u - R(E(u))\|_X,$$

we define the nonlinear n-width of $\mathcal{K}$ as

$$\delta_n(\mathcal{K})_X := \inf_{E,R} \max_{u \in \mathcal{K}} \|u - R(E(u))\|_X,$$

where the infimum is taken over all continuous maps (E, R). Comparison with the Kolmorgorov n-width : $\delta_n \leq d_n$ and sometimes substantially smaller.

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Infinitely smooth functions

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then, for the linear width,

$$\min\{n \ : \ d_n(\mathcal{K})_X \leq 1/2\} \geq c2^{d/2}.$$

High dimensional problems occur frequently :

PDE's with solutions $u(x, v, t)$ defined in phase space : $d = 7$.

Post-processing of numerical codes : u solver with imput parameters $(y_1, \cdots, y_d)$.

Learning theory : u regression function of imput parameters $(y_1, \cdots, y_d)$

In these applications d may be of the order up to 10^3.

Approximation of stochastic-parametric PDEs : $d = +\infty$.

Smoothness properties of functions should be revisited by other means than C^m classes, and appropriate approximation tools should be used.

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Parametric/Stochastic PDEs

We are interested in PDE's of the general form

$$\mathcal{D}(u, y) = 0,$$

where $\mathcal{D}$ is a partial differential operator, u is the unknown and $y = (y_j)_{j=1,\dots,d}$ is a parameter vector of dimension $d >> 1$ or $d = \infty$ ranging in some domain U.

We assume well-posedness of the solution in some Banach space V for every $y \in U$,

$$y \mapsto u(y)$$

is the solution map from U to V.

Solution manifold $\mathcal{M} := \{u(y) \ : \ y \in U\} \subset V$.

The parameters may be deterministic (control, optimization, inverse problems) or random (uncertainty modeling and propagation, risk assessment). In the second case the solution $u(y)$ is a V-valued random variable.

These applications often requires many queries of $u(y)$, and therefore in principle running many times a numerical solver.

Objective : economical numerical approximation of the map $y \mapsto u(y)$.

Related objectives : numerical approximation of scalar quantities of interest $y \mapsto Q(y) = Q(u(y))$, or of averaged quantities $\overline{u} = \mathbb{E}(u(y))$ or $\overline{Q} = \mathbb{E}(Q(y))$.

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Guiding example : elliptic PDEs

We consider the steady state diffusion equation

$$-\mathrm{div}(a\nabla u) = f \ \text{ on } \ D \subset \mathbf{R}^{\mathrm{m}} \ \text{ and } \ \mathrm{u}_{|\partial \mathrm{D}} = 0,$$

set on a domain $D \subset \mathbb{R}^m$, where $f = f(x) \in L^2(D)$ and $a \in L^\infty(D)$

Lax-Milgram lemma : assuming $a_{\min} := \min_{x \in D} a(x) > 0$, unique solution $u \in V = H_0^1(D)$ with

$$\|u\|_V := \|\nabla u\|_{L^2(D)} \leq \frac{1}{a_{\min}} \|f\|_{V'}.$$

Proof of the estimate : multiply equation by u and integrate

$$a_{\min}\|u\|_V^2 \leq \int_D a\nabla u \cdot \nabla u = -\int_D u\,\mathrm{div}(a\nabla u) = \int_D uf \leq \|u\|_V \|f\|_{V'}.$$

We may extend this theory to the solution of the weak (or variational) formulation

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$$a_{\min}\|u\|_V^2 \leq \int_D a\nabla u \cdot \nabla u = -\int_D u\,\mathrm{div}(a\nabla u) = \int_D uf \leq \|u\|_V \|f\|_{V'}.$$

We may extend this theory to the solution of the weak (or variational) formulation

$$\int_D a\nabla u \cdot \nabla v = \langle f, v\rangle, \quad v \in V = H_0^1(D),$$

if $f \in V' = H^{-1}(D)$

Parametrization

Assume diffusion coefficients in the form of an expansion

$$a = a(y) = \overline{a} + \sum_{j \geq 1} y_j \psi_j, \quad y = (y_j)_{j \geq 1} \in U,$$

with $d >> 1$ or $d = \infty$ terms, where $\overline{a}$ and $(\psi_j)_{j \geq 1}$ are functions from L^∞,

Note that $a(y)$ is a function for each given y. We may also write

$$a = a(x, y) = \overline{a}(x) + \sum_{j \geq 1} y_j \psi_j(x), \quad x \in D, y \in U,$$

where x and y are the spatial and parametric variable, respectively. Likewise, the corresponding solution $u(y)$ is a function $x \mapsto u(y, x)$ for each given y. We often ommit the reference to the spatial variable.

Up to a change of variable, we assume that all y_j range in $[-1, 1]$, therefore

$$y \in U = [-1, 1]^d \text{ or } [-1, 1]^{\mathbb{N}}.$$

Uniform ellipticity assumption :

$$(UEA) \qquad 0 < r \leq a(x, y) \leq R, \quad x \in D, \, y \in U$$

Then the solution map is bounded from U to $V := H_0^1(\Omega)$, that is, $u \in L^\infty(U, V)$:

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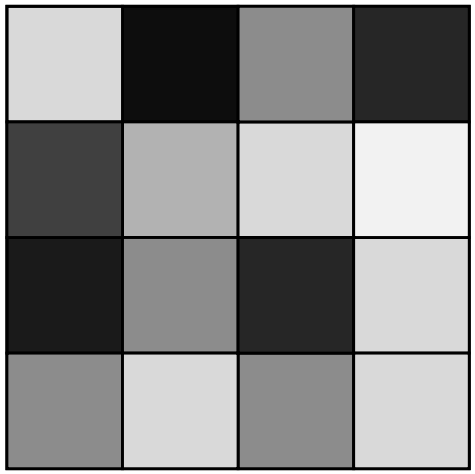
Example of parametrization : piecewise constant coefficients

Assume that a is piecewise constant over a partition $\{D_1, \dots, D_d\}$ of D, and such that on each D_j the value of a varies on $[c - c_j, c + c_j]$ for some $c > 0$ and $0 < c_j < c$.



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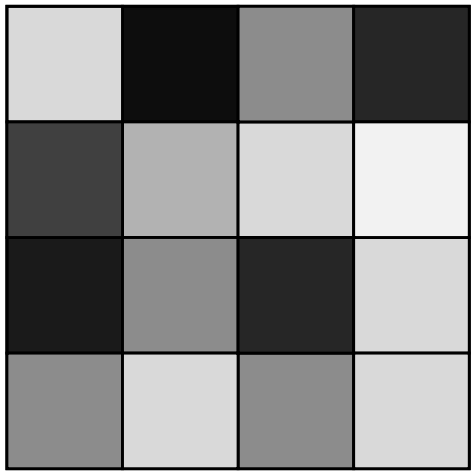

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Assume $a = (a(x))_{x \in D}$ is a random process with average

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$$C_a(x,z) = \mathbb{E}\Big(\tilde{a}(x)\tilde{a}(z)\Big), \quad \tilde{a} := a - \overline{a}, \quad x, z \in D.$$

Define the integral operator by

$$Tv(x) = \int_D C_a(x,z)v(z)dz,$$

self-adjoint, positive and compact in $L^2(D)$. Therefore it admits an L^2 orthonormal basis $(\varphi_j)_{j \geq 1}$ of eigenfunctions, associated to eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, such that $\lambda_n \to 0$ as $n \to +\infty$.

Karhunen-Loeve (KL) decomposition (a.k.a. principal component analysis) :

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Properties of KL representation

The ξ_j are centered and decorelated scalar random variables, with

$$\mathbb{E}(\xi_j) = 0, \quad \mathbb{E}(\xi_i \xi_j) = 0 \quad \text{if} \quad j \neq i, \quad \mathbb{E}(|\xi_j|^2) = \lambda_j.$$

If the random process a is bounded, then the variables ξ_j have bounded range $|\xi_j| \leq c_j$, so that with $y_j := \xi_j / c_j$ and $\psi_j := c_j \varphi_j$ we may also write

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Case of a stationary process : $C_a(x, z) = \kappa(x - z)$, that is T is a convolution operator. If D is the m-dimensional 2π-periodic torus, the KL basis is of Fourier type

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Model reduction

Objective : fast approximate computation of $y \mapsto u(y)$ for many queries of y.

Vehicle : separable (low rank) approximations of the form

$$u(x,y) \approx u_n(x,y) := \sum_{k=1}^{n} v_k(x)\phi_k(y),$$

where $v_k : D \to \mathbb{R}$ with $v_k \in V$ and $\phi_k : U \to \mathbb{R}$. Equivalently

$$u_n(y) := \sum_{k=1}^{n} v_k \phi_k(y) = \sum_{k=1}^{n} \phi_k(y) v_k \in V_n := \mathrm{span}\{v_1, \ldots, v_n\} \subset V, \quad y \in U.$$

Thus we approximate simultaneously all solutions $u(y)$ in the same n-dimensional space $V_n \subset V$.

By the way, this is what we do when we use a finite element solver :

$$y \mapsto u_h(y) \in V_h \subset V.$$

So what's new here ?

Accurate solutions may require V_h of very large dimension $N_h = \dim(V_h) >> 1$ and each query $y \mapsto u_h(y)$ is expensive.

We hope to achieve same order of accuracy $n << N_h$ by a choice of V_n adapted to the parametric problem. In practice the functions $v_1, \ldots, v_n$ are typically picked from such a finite element space V_h, so that $u_n(y) \in V_h$ for all y but actually belongs to the much smaller space $V_n \subset V_h$.

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Measure of performance

1. Uniform sense

$$\|u - u_n\|_{L^\infty(U,V)} := \sup_{y \in U} \|u(y) - u_n(y)\|_V,$$

2. Mean-square sense, for some measure μ on U,

$$\|u - u_n\|^2_{L^2(U,V,d\mu)} := \int_U \|u(y) - u_n(y)\|^2_V d\mu(y).$$

If μ is a probability measure, and y randomly distributed according to this measure, we have

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A "worst case" estimate is always above an "average" estimate.

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Optimal spaces ?

Best n-dimensional space for approximation in the uniform sense : the space F_n one that reaches the Kolmogorov n-width of the solution manifold in the V norm

$$d_n = d_n(\mathcal{M}) := \inf_{\dim(E)\leq n} \sup_{v\in\mathcal{M}} \min_{w\in E} \|v-w\|_V = \inf_{\dim(E)\leq n} \sup_{y\in U} \min_{w\in E} \|u(y)-w\|_V.$$

Best n-dimensional space for approximation in the mean-square sense : principal component analysis in V (instead of L^2 with KL basis). Consider an orthonormal basis $(e_k)_{k\geq 1}$ of V and decompose

$$u(y) := \sum_{k\geq 1} u_k(y) e_k, \quad u_k(y) := \langle u(y), e_k\rangle_V.$$

Introduce the infinite correlation matrix $M = (\mathbb{E}(u_k u_l))_{k,l\geq 1}$. It has eigenvalues $(\lambda_k)_{k\geq 1}$ and associated eigenvectors $g_k = (g_{k,l})_{l\in\mathbb{N}}$ which form an orthonormal basis of $\ell^2(\mathbb{N})$. The best space is

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Realistic strategies

The optimal spaces F_n and G_n are usually out of reach. There are two main computational approaches to realistically design the approximation $u_n = \sum_{k=1}^{n} v_k \phi_k$.

1. Expand formally the solution map $y \mapsto u(y)$ in a given "basis" $(\phi_k)_{k \geq 1}$ of high dimensional functions

$$u(y) = \sum_{k \geq 1} v_k \phi_k(y),$$

where $v_k \in V$ are viewed as the coefficients in this expansion.

Compute these coefficients for $k = 1, \ldots, n$ approximately by some numerical procedure.

Main representative : Polynomial methods (the ϕ_k are multivariate polynomials).

2. Compute first a "good" basis $\{v_1, \ldots, v_n\}$ and define V_n as their span. Then, for any given instance y, compute $u_n(y) \in V_n$ by a numerical method.

Main representative : Reduced Bases (RB) methods emulate the n-width spaces F_n for uniform, or $L^\infty(U, V)$, approximation. Proper Orthogonal Decompositions (POD) methods emulate the principal component spaces G_n for mean-square, or $L^2(U, V, \mu)$, approximation.

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Remarks

In the second approach, the functions v_k are typically computed in an heavy offline stage, then for any given y, the computation of $u_n(y)$ is done in a cheap online stage.

The first approach gives immediate access to the approximation u_n for all values of y since the functions v_k and ϕ_k are both precomputed offline, the online stage is then a trivial recombination.

Other important distinction : intrusive versus non-intrusive methods. The latter are based on post-processing individual solution instances

$$u(y^i), \quad y^i \in U, \quad i = 1, \ldots, m.$$

They may benefit of a pre-existing numerical solver viewed as a blackbox and do not necessarily require full knowledge of PDE model.

In practice, the v_k are typically chosen in a discrete (finite element) space $V_h \subset V$, with $N_h = \dim(V_h) >> n$. Equivalently, we apply the above technique to the discrete solution map $y \mapsto u_h(y) \in V_h$. The error may thus be decomposed into the finite element discretization error and the model reduction error.

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How to defeat the curse of dimensionality ?

The map $y \mapsto u(y)$ is high dimensional, or even infinite dimensional $y = (y_j)_{j \geq 1}$.

We are thus facing the curse of dimensionality when trying to approximate it with conventional discretization tools in the y variable (Fourier series, finite elements).

A possible way out : exploit anisotropic features in the function $y \mapsto u(y)$.

The PDE is parametrized by a function a (diffusion coefficient, velocity, domain boundary) and y_j are the coordinates of a in a certain basis representation $a = \bar{a} + \sum_{j \geq 1} y_j \psi_j$.

If the ψ_j decays as $j \to +\infty$ (for instance if a has some smoothness) then the variable y_j are less active for large j.

We shall see that in certain relevant instances, this mechanism allows to break the curse of dimensionality by using suitable expansions : we obtain approximation rates $\mathcal{O}(N^{-s})$ that are independent of d in the sense that they hold when $d = \infty$.

One key tool for obtaining such result is the concept of sparse approximation.

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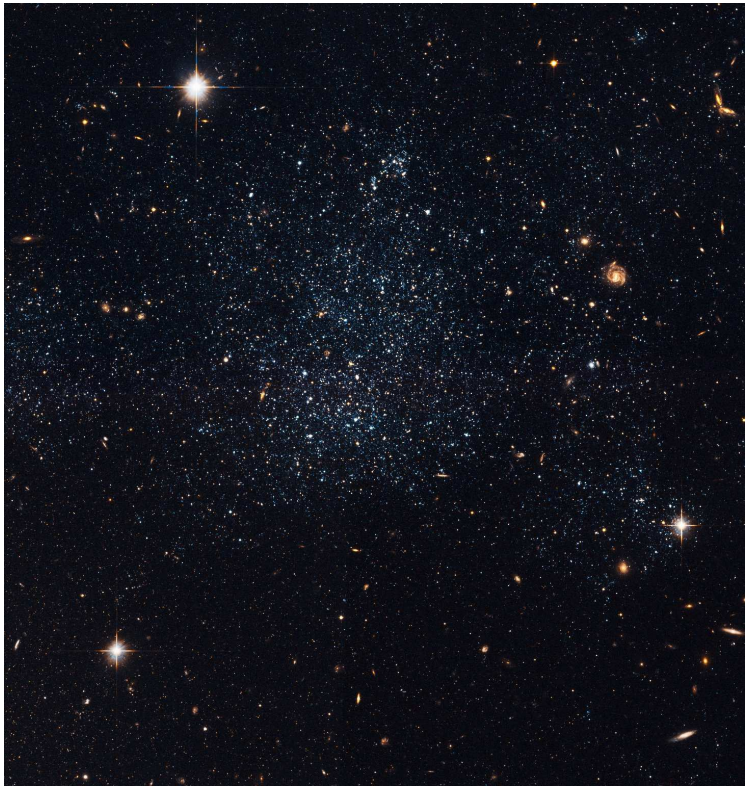
Sparsity

Small dimensional phenomenon in high dimensional context



Simple example : vector $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$ representing a signal, image or function, discretized with $N >> 1$.

The vector x is sparse if only few of its coordinates are non-zero.

How to quantify this ?

The set of n-sparse vectors

$$\Sigma_n := \{x \in \mathbf{R}^N \ ; \ \#\{i \ ; \ x_i \neq 0\} \leq n\}$$

As n gets smaller, $x \in \Sigma_n$ gets sparser.

More realistic : a vector is quasi-sparse if only a few numerically significant coordinates concentrate most of the information. How to measure this notion of concentration ?

Remarks :

A vector in Σ_n is characterized by k non-zero values and their k positions.

Intrinsically nonlinear concepts : $x, y \in \Sigma_n$ does not imply $x + y \in \Sigma_n$.

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Importance of the concept of representation : David Marr ("Vision", Freeman, 1982).

"A representation is a formal system for making explicit certain entities or types of information, together with a specification of how the system does this... For example, the Arabic, Roman and binary numerical systems are all formal systems for representing numbers. The Arabic representation consists in a string of symbols drawn from the set 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and the rule for constructing the description of a particular integer n is that one decomposes n into a sum of multiple of powers of 10...the alphabet allows the construction of a written representation of words... A representation, therefore is not a foreign idea at all, we all use representations all the time. However, the notion that one can capture some aspects of reality by making a description of it using a symbol and that to do so can be useful seems to me a fascinating and powerful idea...
...This issue is important, because how information is presented can greatly affect how easy it is to do different things with it. This is evident even from our number example : it is easy to add, to substract and even to multiply if the Arabic or binary representation are used, but it is not at all easy to do these things - especially multiplication - with Roman numerals. This is a key reason why the Roman culture failed to develop mathematics in the way the Arabic culture had."

The choice of an appropriate representation of a function can be fundamental to solve a specific task.

Fourier representations

- Analysis : $\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$.
- Synthesis : $f(t) = (2\pi)^{-1} \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{i\omega t} d\omega$.

Representation of f in terms of the pure waves $e_\omega(t) = e^{i\omega t}$, $\omega \in \mathbf{R}$.

For 1-periodic functions :

- Analysis : $c_k(f) = \int_0^1 f(t) e^{-i2\pi kt} dt$.
- Synthesis : $f(t) = \sum_{k \in \mathbb{Z}} c_k(f) e^{i2\pi kt}$.

Discrete Fourier transform : $(x[k])_{k=0,\cdots,N-1}$ and $(\hat{x}[k])_{k=0,\cdots,N-1}$ connected by

$$\hat{x}[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-i2\pi nk/N} \quad \text{and} \quad x[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \hat{x}[n] e^{i2\pi nk/N}.$$

Implemented in $\mathcal{O}(N \log N)$ operations by FFT.

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Fourier representations and computation

Approximation of a 1-periodic function by partial sum $S_n f(t) = \sum_{k=-n}^{n} c_k(f) e^{i2\pi kt}$.

Problem : fast convergence ?

If $f, f', \cdots, f^{(m)}$ are continuous over $\mathbf{R}$, we can apply m times the integration by part to obtain

$$\begin{aligned} |c_k(f)| &= |(i2\pi k)^{-1} c_k(f')| \\ &= \cdots |(i2\pi k)^{-m} c_k(f^{(m)})| \\ &\leq |i2\pi k|^{-m} \int_0^1 |f^{(m)}| \leq C_m k^{-m}. \end{aligned}$$

$\Rightarrow$ Fast decay if f is smooth.

However, if f is smooth everywhere except at some discontinuity point $x \in [0,1]$, we cannot hope better than $|c_k(f)| \leq Ck^{-1}$ (also Gibbs phenomenon for $S_n f$ near the singularity).

Better representations are needed for such functions.

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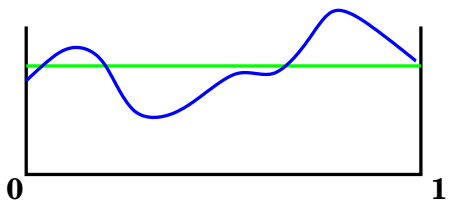
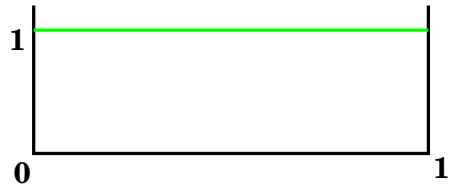
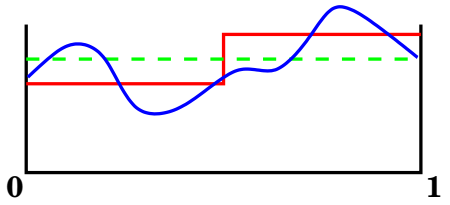
Multiscale representations into wavelet bases : the Haar system



$\mathbf{f = < f , e_0 > e_0}$



$\mathbf{e_0}$



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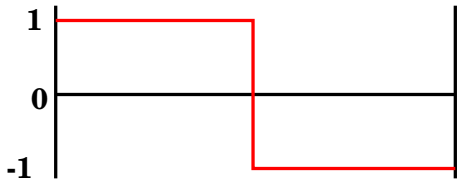

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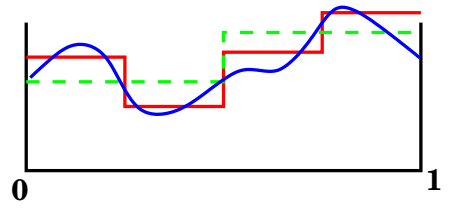

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More general wavelets are constructed from similar multiscale approximation processes, using smoother functions such as splines, finite elements...

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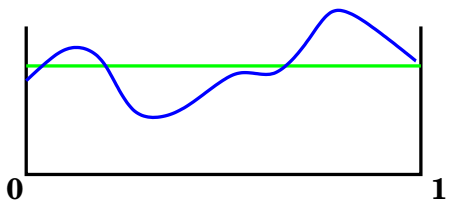

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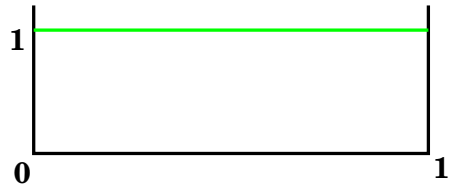

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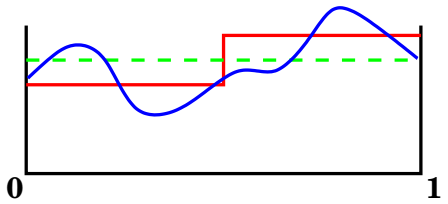

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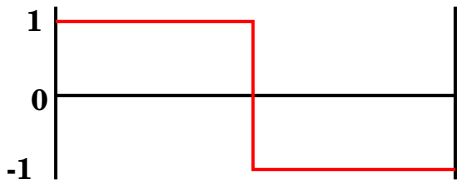

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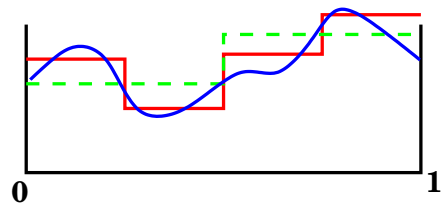

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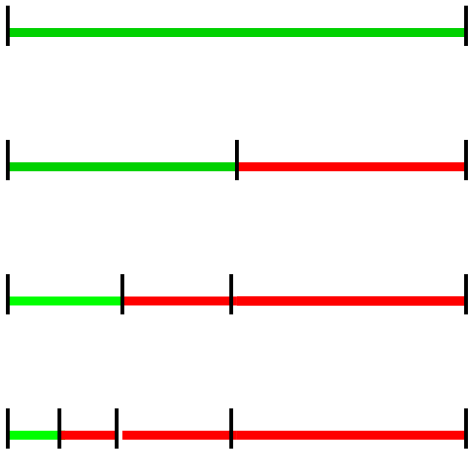
Discrete signals : fast decomposition/reconstruction algorithms



1D array $(f_0, \cdots, f_N)$

$\Rightarrow$ Two half array : averages $\frac{f_{2k}+f_{2k+1}}{2}$ and differences $\frac{f_{2k}-f_{2k+1}}{2}$

$\Rightarrow$ Iterate on the half array of averages...

Multiscale processing of 2D data : separable algorithm

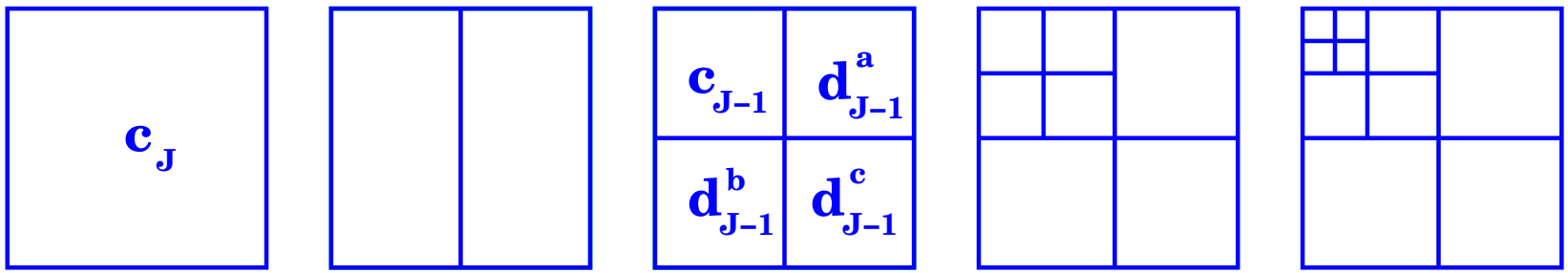


Image $f(k, l) \Rightarrow$ process lines $\Rightarrow$ process columns $\Rightarrow$ Iterate ...

Digital Image 512x512

Multiscale Decomposition

Multiscale decompositions of natural images are sparse : a few numerically significant coefficients concentrate most of the energy and information.

Application to Image Compression

Basic idea : encode with more precision
the few numerically significant coefficients
$\Rightarrow$Resolution is locally adapted
Example : 1 % largest coefficients encoded

Compression standard JPEG 2000 :
- Same basic principles
- Based on smoother wavelets
- Good quality with compression 1/40

Measuring sparsity in a representation $f = \sum f_\lambda \psi_\lambda$

Intuition : growth of number of coefficients above threshold η is controlled as $\eta \to 0$.

Weak spaces : $(f_\lambda) \in w\ell^p$ if and only if

$$\mathrm{Card}\{\lambda \text{ s.t. } |f_\lambda| > \eta\} \leq C\eta^{-p},$$

or equivalently, the decreasing rearrangement $(f_k^*)_{k\geq 1}$ of $(|f_\lambda|)$ satisfies

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The $w\ell^p$ quasi-norm can be defined by

$$\|(f_\lambda)\|_{w\ell^p} := \sup_{k\geq 1} k^{1/p} f_k^*.$$

Obviously $\ell^p \subset w\ell^p$. The representation is sparser as $p \to 0$.

If $p < 2$ and (ψ_λ) is (any) orthonormal basis in a Hilbert space H, an equivalent statement is in terms of best n-term approximation : with $f_n = \sum_{n \text{ largest } |f_\lambda|} f_\lambda \psi_\lambda$,

$$\|f - f_n\|_H = \Big(\sum_{k>n} |f_k^*|^2\Big)^{1/2} \leq \|(f_\lambda)\|_{w\ell^p} \Big(\sum_{k>n} k^{-2/p}\Big)^{1/2} \leq C\|(f_\lambda)\|_{w\ell^p} n^{-s}, \quad s = \frac{1}{p} - \frac{1}{2}.$$

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Measuring sparsity in a representation $f = \sum f_\lambda \psi_\lambda$

Intuition : growth of number of coefficients above threshold η is controlled as $\eta \to 0$.

Weak spaces : $(f_\lambda) \in w\ell^p$ if and only if

$$\mathrm{Card}\{\lambda \text{ s.t. } |f_\lambda| > \eta\} \leq C\eta^{-p},$$

or equivalently, the decreasing rearrangement $(f_k^*)_{k\geq 1}$ of $(|f_\lambda|)$ satisfies

$$f_k^* \leq Ck^{-1/p}.$$

The $w\ell^p$ quasi-norm can be defined by

$$\|(f_\lambda)\|_{w\ell^p} := \sup_{k\geq 1} k^{1/p} f_k^*.$$

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Older observation by Stechkin for the strong ℓ^p spaces

Lemma : one has

$$(f_\lambda)_{\lambda\in\Lambda} \in \ell^p \Rightarrow \|f - f_n\|_H \leq \|(f_\lambda)\|_{\ell^p}(n+1)^{-s}, \quad s = \frac{1}{p} - \frac{1}{2}.$$

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$$(n+1)|f_{n+1}^*|p \leq \sum_{k=1}^{n+1} |f_k^*|^p \leq \|(f_\lambda)\|_{\ell^p}^p.$$

Note that a large value of s corresponds to a value $p < 1$ (non-convex spaces).

For concrete choices of bases a relevant question is thus : what smoothness properties of f ensure that the sequence (f_λ) belongs to ℓ^p or $w\ell^p$ for small values of p ?

In the case of wavelet bases, such properties are characterized by Besov spaces.

In our present setting of high-dimensional functions $y \mapsto u(y)$ we shall rather use tensor-product polynomial bases instead of wavelet bases. Sparsity properties will follow to the anisotropic features of these functions.

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Return to the main guiding example

Steady state diffusion equation

$$-\mathrm{div}(a\nabla u) = f \ \text{ on } \ D \subset \mathbf{R}^m \ \text{ and } \ u_{|\partial D} = 0,$$

where $f = f(x) \in L^2(D)$ and the diffusion coefficients are given by

$$a = a(x,y) = \overline{a}(x) + \sum_{j\geq 1} y_j \psi_j(x),$$

where $\overline{a}$ and the $(\psi_j)_{j\geq 1}$ are given functions and $y \in U := [-1,1]^{\mathbb{N}}$. Uniform ellipticity assumption :

$$(UEA) \qquad 0 < r \leq a(x,y) \leq R, \ \ x \in D, \ y \in U.$$

Equivalent expression of (UEA) : $\bar{a} \in L^\infty(D)$ and

$$\sum_{j\geq 1} |\psi_j(x)| \leq \bar{a}(x) - r, \ \ x \in D,$$

or

$$\left\| \frac{\sum_{j\geq 1} |\psi_j|}{\overline{a}} \right\|_{L^\infty(D)} \leq \theta < 1.$$

Lax-Milgram : solution map is well-defined from U to $V := H_0^1(\Omega)$ with uniform bound

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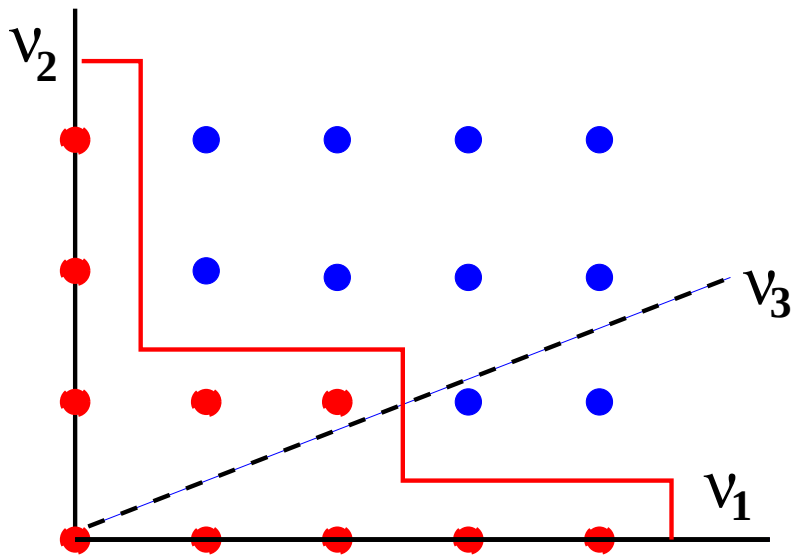
We consider the expansion of $u(y) = \sum_{\nu \in \mathcal{F}} t_\nu y^\nu$, where

$$y^\nu := \prod_{j \geq 1} y_j^{\nu_j} \quad \text{and} \quad t_\nu := \frac{1}{\nu!} \partial^\nu u_{|y=0} \in V \quad \text{with} \quad \nu! := \prod_{j \geq 1} \nu_j! \quad \text{and} \quad 0! := 1.$$

where $\mathcal{F}$ is the set of all finitely supported sequences of integers (finitely many $\nu_j \neq 0$). The sequence $(t_\nu)_{\nu \in \mathcal{F}}$ is indexed by countably many integers.



Objective : identify a set $\Lambda \subset \mathcal{F}$ with $\#(\Lambda) = n$ such that u is well approximated by the partial expansion

$$u_\Lambda(y) := \sum_{\nu \in \Lambda} t_\nu y^\nu.$$

Best n-term approximation

A-priori choices for Λ have been proposed, e.g. (anisotropic) sparse grid defined by restrictions of the type $\sum_j \alpha_j \nu_j \leq A(n)$ or $\prod_j (1+\beta_j \nu_j) \leq B(n)$.

Instead we want to choose Λ optimally adapted to u. By triangle inequality we have

$$\|u - u_\Lambda\|_{L^\infty(U,V)} = \sup_{y\in U} \|u(y) - u_\Lambda(y)\|_V \leq \sup_{y\in U} \sum_{\nu\notin\Lambda} \|t_\nu y^\nu\|_V = \sum_{\nu\notin\Lambda} \|t_\nu\|_V$$

Best n-term approximation in $\ell^1(\mathcal{F})$ norm : use $\Lambda = \Lambda_n$ index set of n largest $\|t_\nu\|_V$.

Lemma : if $(\|t_\nu\|_V)_{\nu\in\mathcal{F}} \in \ell^p(\mathcal{F})$ for some $p < 1$, then for this Λ_n,

$$\sum_{\nu\notin\Lambda_n} \|t_\nu\|_V \leq C n^{-s}, \quad s := \frac{1}{p} - 1, \quad C := \|(\|t_\nu\|_V)\|_p.$$

Proof : with $(t_k^*)_{k>0}$ the decreasing rearrangement, we combine

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One main result

Theorem (Cohen-DeVore-Schwab, 2011) : under the uniform ellipticity assumption (UAE), then for any $p < 1$,

$$(\|\psi_j\|_{L^\infty})_{j>0} \in \ell^p(\mathbb{N}) \Rightarrow (\|t_\nu\|_V)_{\nu\in\mathcal{F}} \in \ell^p(\mathcal{F}).$$

Interpretations :

(i) The Taylor expansion of $u(y)$ inherits the sparsity properties of the expansion of $a(y)$ into the ψ_j.

(ii) We approximate $u(y)$ in $L^\infty(U, V)$ with algebraic rate $\mathcal{O}(n^{-s})$ despite the curse of (infinite) dimensionality, due to the fact that y_j is less influencial as j gets large.

(iii) The solution manifold $\mathcal{M} := \{u(y) \ ; \ y \in U\}$ is uniformly well approximated by the n-dimensional space $V_n := \mathrm{span}\{t_\nu \ : \ \nu \in \Lambda_n\}$. Its n-width satisfies the bound

$$d_n(\mathcal{M})_V \leq \max_{y\in U} \mathrm{dist}(u(y), V_n)_V \leq \max_{y\in U} \|u(y) - u_{\Lambda_n}(y)\|_V \leq Cn^{-s}.$$

Such approximation rates cannot be proved for the usual a-priori choices of Λ.

Same result for more general linear equations $Au = f$ with affine operator dependance : $A = \overline{A} + \sum_{j\geq 1} y_j A_j$ uniformly invertible over $y \in U$, and $(\|A_j\|_{V\to W})_{j\geq 1} \in \ell^p(\mathbb{N})$, as well as other models.

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Idea of proof : extension to complex variable

Estimates on $\|t_\nu\|_V$ by complex analysis : extend $u(y)$ to $u(z)$ with $z = (z_j) \in \mathbb{C}^{\mathbf{N}}$.

Uniform ellipticity $\sum_{j\geq 1} |\psi_j| \leq \overline{a} - r$ implies that with $a(z) = \overline{a} + \sum_{j\geq 1} z_j \psi_j$,

$$0 < r \leq \Re(a(x,z)) \leq |a(x,z)| \leq 2R, \quad x \in D,$$

for all $z \in \mathcal{U} := \{|z| \leq 1\}^{\mathbb{N}} = \otimes\{|z_j| \leq 1\}$.

Lax-Milgram theory applies : $\|u(z)\|_V \leq C_0 = \frac{\|f\|_{V^*}}{r}$ for all $z \in \mathcal{U}$.

The function $u \mapsto u(z)$ is holomorphic in each variable z_j at any $z \in \mathcal{U}$: its first derivative $\partial_{z_j} u(z)$ is the unique solution to

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Note that ∇ is with respect to spatial variable $x \in D$.
Extended domains of holomorphy : if $\rho = (\rho_j)_{j\geq 0}$ is any positive sequence such that for some $\delta > 0$

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$$\int_D a(z) \nabla \partial_{z_j} u(z) \cdot \nabla v = -\int_D \psi_j \nabla u(z) \cdot \nabla v, \quad v \in V.$$

Note that ∇ is with respect to spatial variable $x \in D$.
Extended domains of holomorphy : if $\rho = (\rho_j)_{j\geq 0}$ is any positive sequence such that for some $\delta > 0$

$$\sum_{j\geq 1} \rho_j |\psi_j(x)| \leq \overline{a}(x) - \delta, \quad x \in D,$$

then u is holomorphic with uniform bound $\|u(z)\| \leq C_\delta = \frac{\|f\|_{V^*}}{\delta}$ in the polydisc

$$\mathcal{U}_\rho := \otimes\{|z_j| \leq \rho_j\},$$

If $\delta < r$, we can take $\rho_j > 1$.

Estimate on the Taylor coefficients

Use Cauchy formula. In 1 complex variable if $z \mapsto u(z)$ is holomorphic and bounded in a neighbourhood of disc $\{|z| \leq b\}$, then for all z in this disc

$$u(z) = \frac{1}{2i\pi} \int_{|z'|=b} \frac{u(z')}{z - z'} dz',$$

which leads by n differentiation at $z = 0$ to $|u^{(n)}(0)| \leq n! b^{-n} \max_{|z| \leq b} |u(z)|$.

Recursive application of this to all variables z_j such that $\nu_j \neq 0$, with $b = \rho_j$ gives

$$\|\partial^\nu u_{|z=0}\|_V \leq C_\delta \nu! \prod_{j \geq 1} \rho_j^{-\nu_j},$$

and thus

$$\|t_\nu\|_V \leq C_\delta \prod_{j \geq 1} \rho_j^{-\nu_j} = C_\delta \rho^{-\nu},$$

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Since ρ is not fixed we have

$$\|t_\nu\|_V \leq C_\delta \inf\Big\{\rho^{-\nu} \quad : \quad \rho \ \text{ s.t. } \ \sum_{j\geq 1} \rho_j |\psi_j(x)| \leq \overline{a}(x) - \delta, \ \ x \in D\Big\}.$$

We do not know the general solution to this problem, except in particular case, for example when the ψ_j have disjoint supports.

Instead design a particular choice $\rho = \rho(\nu)$ satisfying the constraint with $\delta = r/2$, for which we prove that

$$(\|\psi_j\|_{L^\infty})_{j\geq 1} \in \ell^p(\mathbb{N}) \Rightarrow (\rho(\nu)^{-\nu})_{\nu\in\mathcal{F}} \in \ell^p(\mathcal{F}),$$

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A simple case

Assume that the ψ_j have disjoint supports. Then we maximize separately the ρ_j so that

$$\sum_{j\geq 1} \rho_j |\psi_j(x)| \leq \overline{a}(x) - \frac{r}{2}, \quad x \in D,$$

which leads to

$$\rho_j := \min_{x\in D} \frac{\overline{a}(x) - \frac{r}{2}}{|\psi_j(x)|}.$$

We have, with $\delta = \frac{r}{2}$,

$$\|t_\nu\|_V \leq C_\delta \rho^{-\nu} = C_\delta b^\nu,$$

where $b = (b_j)$ and

$$b_j := \rho_j^{-1} = \frac{|\psi_j(x)|}{\overline{a}(x) - \frac{r}{2}} \leq \frac{\|\psi_j\|_{L^\infty}}{R - \frac{r}{2}}.$$

Therefore $b \in \ell^p(\mathbb{N})$. From (UEA), we have $|\psi_j(x)| \leq \overline{a}(x) - r$ and thus $\|b\|_{\ell^\infty} < 1$. We finally observe that

$$b \in \ell^p(\mathbb{N}) \text{ and } \|b\|_{\ell^\infty} < 1 \Leftrightarrow (b^\nu)_{\nu\in\mathcal{F}} \in \ell^p(\mathcal{F}).$$

Proof : factorize

$$\sum_{\nu\in\mathcal{F}} b^{p\nu} = \prod_{j\geq 1} \sum_{n\geq 0} b_j^{pn} = \prod_{j\geq 1} \frac{1}{1 - b_j^p}.$$

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Improved summability results

Improved results can be obtained, without relying on complex variable, by better exploiting the specific structure of the diffusion equation.

Recursive formula for the Taylor coefficients : with $e_j = (0, \dots, 0, 1, 0, \dots)$ the Kroeneker sequence of index j, the coefficient t_ν is solution to

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We introduce the quantities

$$d_\nu := \int_D \bar{a} |\nabla t_\nu|^2 \quad \text{and} \quad d_{\nu,j} := \int_D |\psi_j| |\nabla t_\nu|^2.$$

Recall that (UEA) implies that $\left\| \frac{\sum_{j \geq 1} |\psi_j|}{\bar{a}} \right\|_{L^\infty(D)} \leq \theta < 1$. In particular

$$\sum_{j \geq 1} d_{\nu,j} \leq \theta d_\nu.$$

We use here the equivalent norm $\|v\|_V^2 := \int_D \bar{a} |\nabla v|^2$.

Lemma : under (UEA), one has $\sum_{\nu \in \mathcal{F}} d_\nu = \sum_{\nu \in \mathcal{F}} \|t_\nu\|_V^2 < \infty$.

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$$d_\nu = \int_D \bar{a}|\nabla t_\nu|^2 = -\sum_{j:\,\nu_j \neq 0} \int_D \psi_j \nabla t_{\nu - e_j} \nabla t_\nu .$$

Apply Young's inequality on the right side gives

$$d_\nu \leq \sum_{j:\,\nu_j \neq 0} \left(\frac{1}{2} \int_D |\psi_j| |\nabla t_{\nu - e_j}|^2 + \frac{1}{2} \int_D |\psi_j| |\nabla t_\nu|^2 \right) = \frac{1}{2} \sum_{j:\,\nu_j \neq 0} d_{\nu,j} + \frac{1}{2} \sum_{j:\,\nu_j \neq 0} d_{\nu - e_j, j}.$$

The first sum is bounded by θd_ν, therefore

$$\left(1 - \frac{\theta}{2}\right) d_\nu \leq \frac{1}{2} \sum_{j:\,\nu_j \neq 0} d_{\nu - e_j, j}.$$

Now summing over all $|\nu| = k$ gives

$$\left(1 - \frac{\theta}{2}\right) \sum_{|\nu|=k} d_\nu \leq \frac{1}{2} \sum_{|\nu|=k} \sum_{j:\,\nu_j \neq 0} d_{\nu - e_j, j} = \frac{1}{2} \sum_{|\nu|=k-1} \sum_{j \geq 1} d_{\nu,j} \leq \frac{\theta}{2} \sum_{|\nu|=k-1} d_\nu .$$

Therefore $\sum_{|\nu|=k} d_\nu \leq \kappa \sum_{|\nu|=k-1} d_\nu$ with $\kappa := \frac{\theta}{2-\theta} < 1$, and thus $\sum_{\nu \in \mathcal{F}} d_\nu < \infty$.

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Rescaling

Now let $\rho = (\rho_j)_{j\geq 1}$ be any sequence with $\rho_j > 1$ such that $\sum_{j\geq 1} \rho_j |\psi_j| \leq \overline{a} - \delta$ for some $\delta > 0$, or equivalently such that $\left\| \frac{\sum_{j\geq 1} \rho_j |\psi_j|}{\overline{a}} \right\|_{L^\infty(D)} \leq \theta < 1$.

Considered the rescaled solution map $\tilde{u}(y) = u(\rho y)$ where $\rho y := (\rho_j y_j)_{j\geq 1}$ which is the solution of the same problem as u with ψ_j replaced by $\rho_j \psi_j$.

Since (UEA) holds for for these rescaled functions, the previous lemma shows that

$$\sum_{\nu \in \mathcal{F}} \|\tilde{t}_\nu\|_V^2 < \infty,$$

where

$$\tilde{t}_\nu := \frac{1}{\nu!} \partial^\nu \tilde{u}(0) = \frac{1}{\nu!} \rho^\nu \partial^\nu u(0) = \rho^\nu t_\nu.$$

This therefore gives the weighted ℓ^2 estimate

$$\sum_{\nu \in \mathcal{F}} (\rho^\nu \|t_\nu\|_V)^2 \leq C < \infty.$$

In particular, we retrieve the estimate $\|t_\nu\|_V \leq C\rho^{-\nu}$ that was obtained by the complex variable approach, however the above estimate is stronger.

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An alternate summaibility result

Applying Hölder's inequality gives

$$\sum_{\nu\in\mathcal{F}} \|t_\nu\|_V^p \le \Big(\sum_{\nu\in\mathcal{F}} (\rho^\nu \|t_\nu\|_V)^2\Big)^{p/2} \Big(\sum_{\nu\in\mathcal{F}} \rho^{-q\nu}\Big)^{1-p/2},$$

with $q = \frac{2p}{2-p} > p$, or equivalently $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$.

The sum in second factor is finite provided that $(\rho_j^{-1})_{j\ge 1} \in \ell^q$. Therefore, the following result holds.

Theorem (Bachmayr-Cohen-Migliorati, 2015) : Let p and q be such that $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$. Assume that there exists a sequence $\rho = (\rho_j)_{j\ge 1}$ of numbers larger than 1 such that

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No improvement in the case of globally supported functions, such as typical KL bases.

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Similar improved results if the ψ_j have supports with limited overlap, such as wavelets.

No improvement in the case of globally supported functions, such as typical KL bases.

Other models

Model 1 : same PDE but no affine dependence, e.g. $a(x,y) = \overline{a}(x) + (\sum_{j\geq 0} y_j \psi_j(x))^2$. Assuming that $\overline{a}(x) \geq r > 0$ guarantees ellipticity uniformly over $y \in U$.

Model 2 : similar problems + non-linearities, e.g.

$$g(u) - \operatorname{div}(a\nabla u) = f \ \text{ on } \ D = D(y) \ \ u_{|\partial D} = 0,$$

with same assumptions on a and f. Well-posedness in $V = H_0^1(D)$ for all $f \in V'$ is ensured for certain nonlinearities, e.g. $g(u) = u^3$ of u^5 in dimension $m = 3$ ($V \subset L^6$).

Model 3 : PDE's on domains with parametrized boundaries, e.g.

$$-\Delta v = f \ \text{ on } \ D = D_y \ \ u_{|\partial D} = 0.$$

where the boundary of D_y is parametrized by y, e.g.

$$D_y := \{(x_1, x_2) \in \mathbb{R}^2 \ : \ 0 < x_1 < 1 \ \text{ and } \ 0 < x_2 < b(x_1, y)\},$$

where $b = b(x,y) = \overline{b}(x) + \sum_j y_j \psi_j(x)$ satisfies $0 < r < b(x,y) < R$. We transport this problem on the reference domain $[0,1]^2$ and study

$$u(y) := v(y) \circ \phi_y, \quad \phi_y : [0,1]^2 \to D_y, \quad \phi_y(x_1, x_2) := (x_1, x_2 b(x_1, y)).$$

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Polynomial approximation for these models

In contrast to our guiding example (which we refer to as model 0), bounded holomorphic extension is generally not feasible in a complex domain containing the polydisc $\mathcal{U} = \otimes\{|z_j| \leq 1\}$. For this reason, Taylor series are not expected to converge.

Instead we consider the tensorized Legendre expansion

$$u(y) = \sum_{\nu \in \mathcal{F}} v_\nu L_\nu(y),$$

where $L_\nu(y) := \prod_{j \geq 1} L_{\nu_j}(y_j)$ and $(L_k)_{k \geq 0}$ are the Legendre polynomials normalized in $L^2\left([-1,1], \frac{dt}{2}\right)$.

Thus $(L^\nu)_{\nu \in \mathcal{F}}$ is an orthonormal basis for $L^2(U, V, \mu)$ where $\mu := \otimes_{j \geq 1} \frac{dy_j}{2}$ is the uniform probability measure and we have

$$v_\nu = \int_U u(y) L_\nu(y) d\mu(y).$$

We also consider the L^∞-normalized Legendre polynomials $P_k = (1+2k)^{-1/2} L_k$ and their tensorized version P_ν, so

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Main result

Theorem (Chkifa-Cohen-Schwab, 2013) : For models 0, 1, 2 and 3, and for any $p < 1$,

$$(\|\psi_j\|_X)_{j>0} \in \ell^p(\mathbb{N}) \Rightarrow (\|v_\nu\|_V)_{\nu\in\mathcal{F}} \text{ and } (\|w_\nu\|_V)_{\nu\in\mathcal{F}} \in \ell^p(\mathcal{F}).$$

with $X = L^\infty$ for models 0, 1, 2, and $X = W^{1,\infty}$ for model 3.

By the same application of Stechkin's argument as for Taylor expansions, best n-term truncations for the L^∞ normalized expansion converge rate $\mathcal{O}(n^{-s})$ in $L^\infty(U, V)$ where $s = \frac{1}{p} - 1$.

Best n-term truncations for the L^2 normalized expansion converge rate $\mathcal{O}(n^{-r})$ in $L^2 U, V, \mu)$ where $r = \frac{1}{p} - \frac{1}{2}$.

In the particular case of our guiding example, model 0, we can obtain improved summability results for Legendre expansions, similar to Taylor expansions.

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Key ingredient in the proof of the above theorem : estimates of Legendre coefficients for holomorphic functions in a "small" complex neighbourhood of U.

Taylor vs Legendre expansions

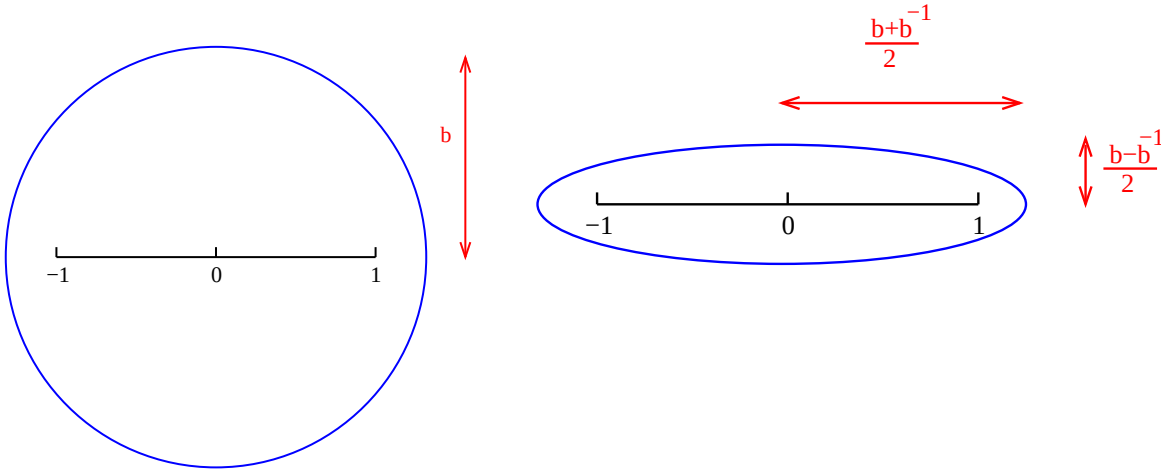
In one variable :

- If u is holomorphic in an open neighbourhood of the disc $\{|z| \leq b\}$ and bounded by M on this disc, then the n-th Taylor coefficient of u is bounded by

$$|t_n| := \left| \frac{u^{(n)}(0)}{n!} \right| \leq M b^{-n}$$

- If u is holomorphic in an open neighbourhood of the domain $\mathcal{E}_b$ limited by the ellipse of semi axes of length $(b + b^{-1})/2$ and $(b - b^{-1})/2$, for some $b > 1$, and bounded by M on this domain, then the n-th Legendre coefficent w_n of u is bounded by

$$|w_n| \leq M b^{-n}(1 + 2n)\phi(b), \qquad \phi(b) := \frac{\pi b}{b - 1}$$



A general assumption for sparsity of Legendre expansions

We say that the solution to a parametric PDE $\mathcal{D}(u, y) = 0$ satisfies the (p, ε)-holomorphy property if and only if there exist a sequence $(c_j)_{j\geq 1} \in \ell^p(\mathbb{N})$, a constant $\varepsilon > 0$ and $C_0 > 0$, such that : for any sequence $\rho = (\rho_j)_{j\geq 1}$ such that $\rho_j > 1$ and

$$\sum_{j\geq 1} (\rho_j - 1) c_j \leq \varepsilon,$$

the solution map has a complex extension

$$z \mapsto u(z),$$

of the solution map that is holomorphic with respect to each variable on a domain of the form $\mathcal{O}_\rho = \otimes_{j\geq 1} \mathcal{O}_{\rho_j}$ where $\mathcal{O}_{\rho_j}$ is an open neigbourhood of the elliptical domain $\mathcal{E}_{\rho_j}$, with bound

$$\sup_{z\in\mathcal{E}_\rho} \|u(z)\|_V \leq C_0,$$

where $\mathcal{E}_\rho = \otimes_{j\geq 1} \mathcal{E}_{\rho_j}$.

Under such an assumption, one has (up to additional harmless factors) an estimate of the form

$$\|w_\nu\|_V \leq C_0 \inf\left\{\rho^{-\nu} \;;\; \rho \text{ s.t. } \sum_{j\geq 1} (\rho_j - 1) c_j \leq \varepsilon\right\},$$

allowing us to prove that $(\|w_\nu\|_V)_{\nu\in\mathcal{F}} \in \ell^p(\mathcal{F})$.

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A general framework for establishing the (p, ε)-holomorphy assumption

Assume a general problem of the form

$$\mathcal{P}(u, a) = 0,$$

with $a = a(y) = \overline{a} + \sum_{j \geq 1} y_j \psi_j$, where

$$\mathcal{P} : V \times X \to W,$$

with V, X, W a triplet of complex Banach spaces, and $\overline{a}$ and ψ_j are functions in X.

Theorem (Chkifa-Cohen-Schwab, 2013) : assume that

(i) The problem is well posed for all $a \in Q = a(U)$ with solution $u(y) = u(a(y)) \in V$.

(ii) The map $\mathcal{P}$ is differentiable (holomorphic) from $X \times V$ to W.

(iii) For any $a \in Q$, the differential $\partial_u \mathcal{P}(u(a), a)$ is an isomorphism from V to W

(iv) One has $(\|\psi_j\|_X)_{j \geq 1}$ in $\ell^p(\mathbb{N})$ for some $0 < p < 1$,

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Idea of proof

Based on the holomorphic Banach valued version of the implicit function theorem (see e.g. Dieudonné).

1. For any $a \in Q = \{a(y) \ : \ y \in U\}$ we can find a $\varepsilon_a > 0$ such that the map $a \longrightarrow u(a)$ has an holomorphic extension on the ball $B(a, \varepsilon_a) := \{\tilde{a} \in X \ : \ \|\tilde{a} - a\|_X < \varepsilon_a\}$.

2. Using the decay properties of the $\|\psi_j\|_X$, we find that Q is compact in X. It can be covered by a finite union of balls $B(a_i, \varepsilon_{a_i})$, for $i = 1, \ldots, M$.

3. Thus $a \longrightarrow u(a)$ has an holomorphic extension on a complex neighbourhood $\mathcal{N}$ of Q of the form

$$\mathcal{N} = \cup_{i=1}^{M} B(a_i, \varepsilon_{a_i}).$$

4. For ε small enough, one proves that if $\sum_{j \geq 1} (\rho_j - 1) c_j \leq \varepsilon$ with $c_j := \|\psi_j\|_L$ then with $\mathcal{O}_\rho = \otimes_{j \geq 1} \mathcal{O}_{\rho_j}$ where $\mathcal{O}_b := \{z \in \mathbb{C} \ : \ \mathrm{dist}(z, [-1, 1])_{\mathbb{C}} \leq b - 1\}$ is a neighborhood of $\mathcal{E}_b$, one has

$$z \in \mathcal{O}_\rho \Rightarrow a(z) \in \mathcal{N}.$$

This gives holomorphy of $z \mapsto u(z) = u(a(z))$ in each variable for $z \in \mathcal{O}_\rho$.

Lognormal coefficients

We assume diffusion coefficients are given by

$$a = \exp(b),$$

with b a random function defined by an affine expansion of the form

$$b = b(y) = \sum_{j \geq 1} y_j \psi_j,$$

where (ψ_j) is a given family of functions from $L^\infty(D)$ and $y = (y_j)_{j \geq 1}$ a sequence of i.i.d. standard Gaussians $\mathcal{N}(0,1)$ variables.

Thus y ranges in $U = \mathbb{R}^{\mathbb{N}}$ equipped with the probabilistic structure $(U, \mathcal{B}(U), \gamma)$ where $\mathcal{B}(U)$ is the cylindrical Borel Σ-algebra and γ the tensorized Gaussian measure.

Commonly used stochastic model for diffusion in porous media.

The solution $u(y)$ is well defined in V for those $y \in U$ such that $b(y) \in L^\infty(D)$, with

$$\|u(y)\|_V \leq \frac{1}{a_{\min}(y)} \|f\|_{V'} \leq \exp(\|b(y)\|_{L^\infty}) \|f\|_{V'}.$$

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Affine Gaussian representations

Given a centered Gaussian process $(b(x))_{x\in D}$ with covariance function $C_b(x,z) = \mathbb{E}(b(x)b(z))$, one frequently consider the Karhunen-Loeve expansion,

$$b = \sum_{j\geq 1} \xi_j \varphi_j,$$

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Example : b the Brownian bridge on $D = [0,1]$ defined by $C_b(x,z) := \min\{x,z\} - xz$.

1. Normalized KL : $\psi_j(x) = \frac{\sqrt{2}}{\pi j}\sin(\pi j x)$.

2. Levy-Ciesielski representation : uses Schauder basis (primitives of Haar system)

$$\psi_{l,k}(x) := 2^{-l/2}\psi(2^l x - k), \quad k = 0,\dots,2^l-1, \quad l \geq 0, \quad \psi(x) := \frac{1}{2}(1-|2x-1|)_+.$$

Then with coarse to fine ordering $\psi_j = \psi_{l,k}$ for $j = 2^l + k$, one has $b = \sum_{j\geq 1} y_j \psi_j$.

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2. Levy-Ciesielski representation : uses Schauder basis (primitives of Haar system)

$$\psi_{l,k}(x) := 2^{-l/2}\psi(2^l x - k), \quad k = 0,\dots,2^l-1, \quad l \geq 0, \quad \psi(x) := \frac{1}{2}(1-|2x-1|)_+.$$

Then with coarse to fine ordering $\psi_j = \psi_{l,k}$ for $j = 2^l + k$, one has $b = \sum_{j\geq 1} y_j \psi_j$.

Affine Gaussian representations

Given a centered Gaussian process $(b(x))_{x\in D}$ with covariance function $C_b(x,z) = \mathbb{E}(b(x)b(z))$, one frequently consider the Karhunen-Loeve expansion,

$$b = \sum_{j\geq 1} \xi_j \varphi_j,$$

where ξ_j are i.i.d. $\mathcal{N}(0,\sigma_j^2)$ and $(\varphi_j)_{j\geq 1}$ are $L^2(D)$-orthonormal, and normalize

$$\psi_j = \sigma_j \varphi_j \quad \text{and} \quad y_j = \sigma_j^{-1}\xi_j,$$

so that $b = \sum_{j\geq 1} y_j\psi_j$. However, other representations may be relevant.

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Main theoretical questions

1. Integrability : under which conditions is $y \mapsto u(y)$ Bochner measurable with values in V and satifies for $0 \leq k < \infty$.

$$\|u\|^k_{L^k(U,V,\gamma)} = \mathbb{E}(\|u(y)\|^k_V) < \infty,$$

In view of $\|u(y)\|_V \leq \exp(\|b(y)\|_{L^\infty})\|f\|_{V'}$, this holds if $\mathbb{E}(\exp(k\|b(y)\|_{L^\infty}) < \infty$.

2. Approximability : if $u \in L^2(U,V,\gamma)$, consider the multivariate Hermite expansion

$$u = \sum_{\nu \in \mathcal{F}} u_\nu H_\nu, \quad H_\nu(y) := \prod_{j \geq 1} H_{\nu_j}(y_j) \quad \text{and} \quad u_\nu := \int_U u(y) H_\nu(y) d\gamma(y)$$

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Best n-term approximation : $u_n = \sum_{\nu \in \Lambda_n} u_\nu H_\nu$, with Λ_n indices of n largest $\|u_\nu\|_V$.

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$$\|u - u_n\|_{L^2(U,V,\gamma)} \leq C n^{-s}, \quad s := \frac{1}{p} - \frac{1}{2}, \quad C := \|(\|u_\nu\|_V)_{\nu \in \mathcal{F}}\|_{\ell^p}$$

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Existing results

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Theorem (Bachmayr-Cohen-DeVore-Migliorati, 2015) :

Let $0 < p < 2$ and define $q := q(p) = \frac{2p}{2-p} > p$ (or equivalently $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$).

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Similar result for the Taylor and Legendre coefficients for the affine parametric model $a(y) = \overline{a} + \sum_{j\geq 1} y_j \psi_j$ however by different arguments.

Proof is rather specific to the linear diffusion equation (yet extensions possible).

Our conditions for ℓ^p summability of $(\|u_\nu\|_V)_{\nu\in\mathcal{F}}$ are weaker than ℓ^p summability of $(j\|\psi_j\|_{L^\infty})_{j\geq 1}$ especially for locally supported ψ_j.

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KL representation :

Globally supported functions $\psi_j(x) = \frac{\sqrt{2}}{\pi j} \sin(\pi j x)$.

The decay of $(\|\psi_j\|_{L^\infty})_{j \geq 1}$ is not sufficient to apply our results.

No provable approximability by best n-term Hermite series.

Schauder representation :

Wavelet type functions with decay in scale $\|\psi_\lambda\|_{L^\infty} \sim 2^{-l/2}$.

This allows to apply our result $\rho_\lambda = 2^{\beta l}$, for any $\beta < \frac{1}{2}$.

Our result imply that $(\|u_v\|_V)_{v \in \mathcal{F}} \in \ell^p(\mathcal{F})$ for any p such that $\frac{1}{2} > \frac{1}{p} - \frac{1}{2}$.

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From approximation results to numerical methods

The results so far are approximation results. They say that for several models of parametric PDEs, the solution map $y \mapsto u(y)$ can be accurately approximate (with rate n^{-s} for some $s > 0$) by multivariate polynomials having n terms.

These polynomials are computed by best n-term truncation of Taylor or Legendre or Hermite series, but this is not feasible in practical numercial methods.

Problem 1 : the best n-term index sets Λ_n are computationally out of reach. Their identification would require the knowledge of all coefficients in the expansion.

Objective : identify non-optimal yet good sets Λ_n.

Problem 2 : the exact polynomial coefficients t_ν (or v_ν, w_ν, u_ν) of u for the indices $\nu \in \Lambda_n$ cannot be computed exactly.

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Numerical methods : strategies to build the sets Λ_n

(i) Non-adaptive, based on the available a-priori estimates for the $\|t_\nu\|_V$ (or $\|v_\nu\|_V$, $\|w_\nu\|_V$, $\|u_\nu\|_V$). Take Λ_n to be the set corresponding to the n largest such estimates.

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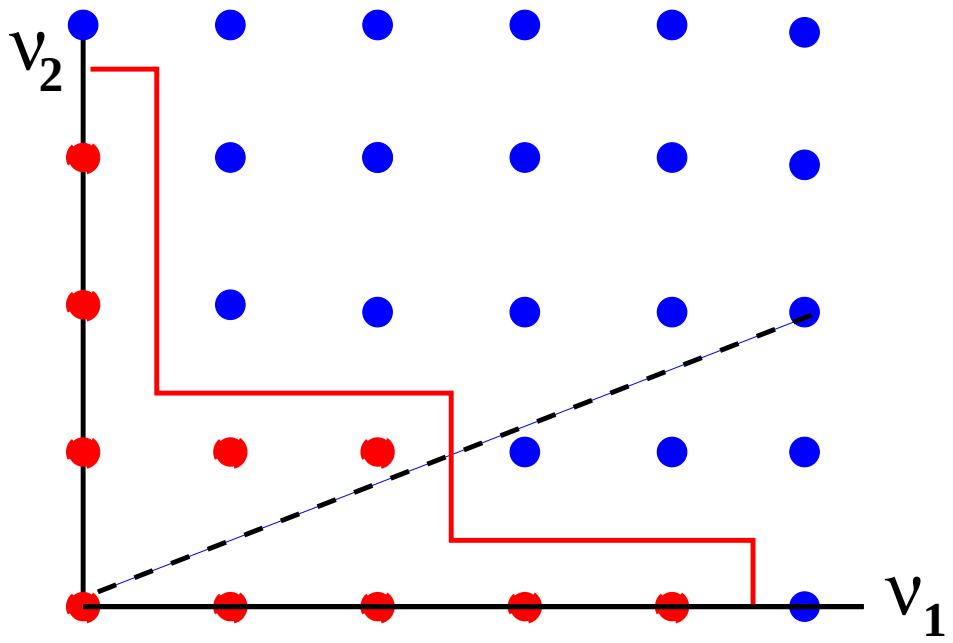

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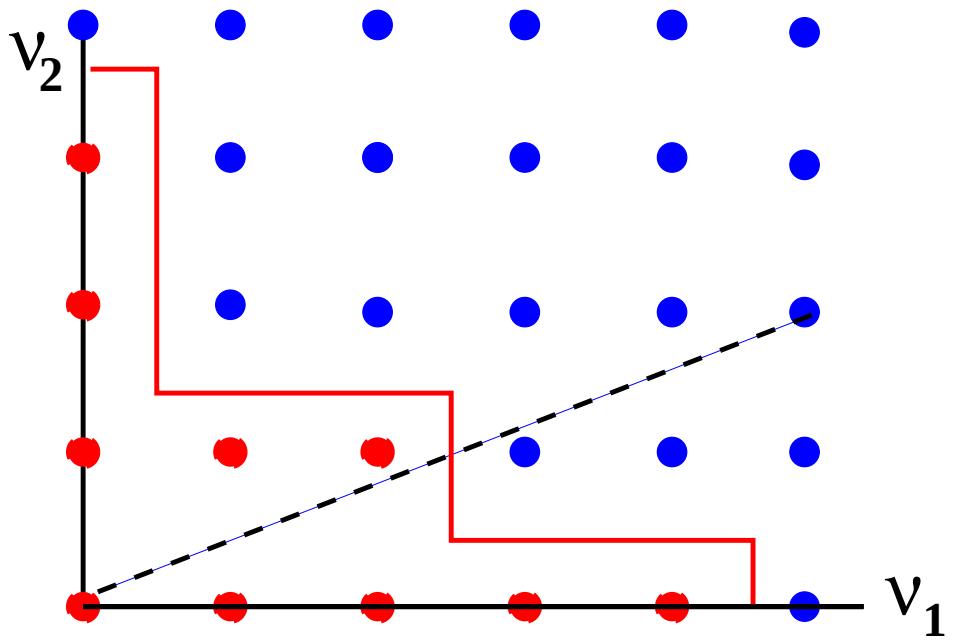

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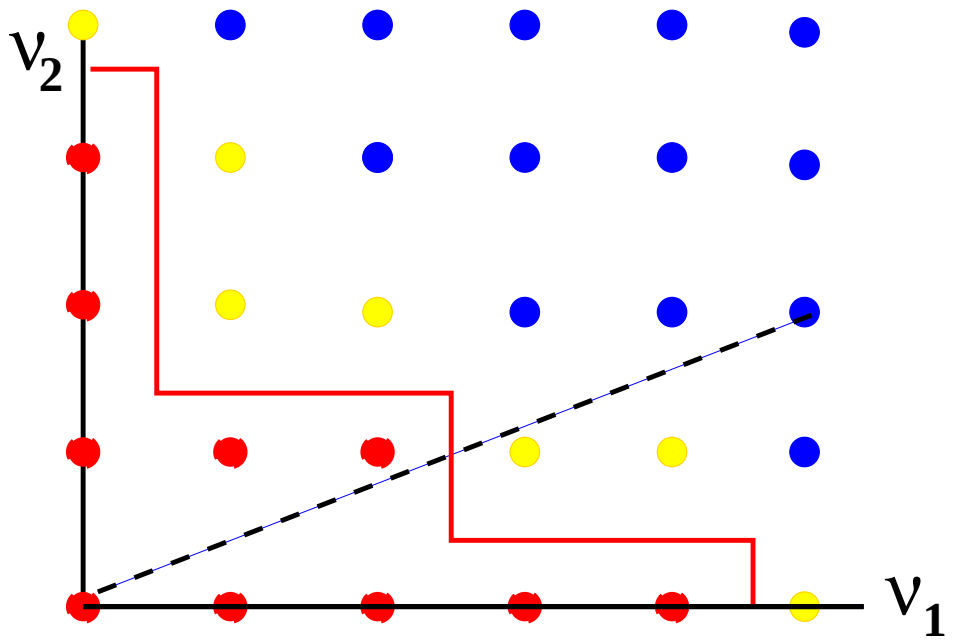

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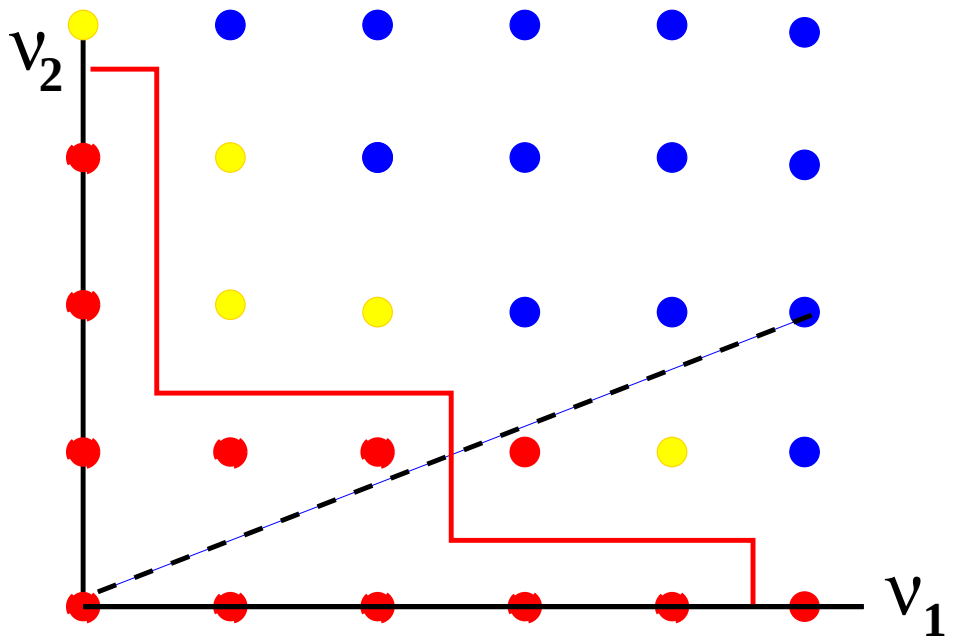

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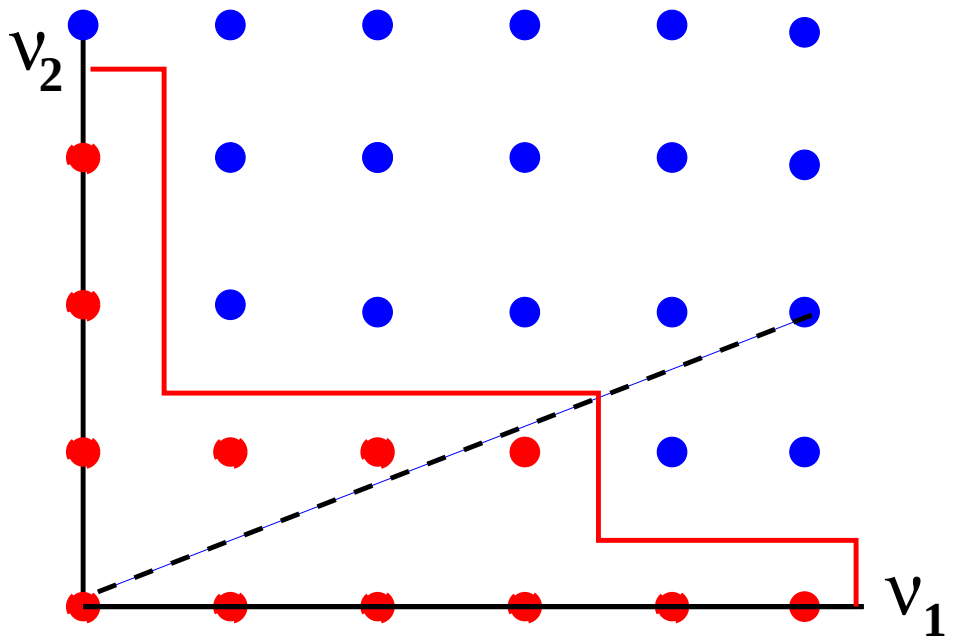
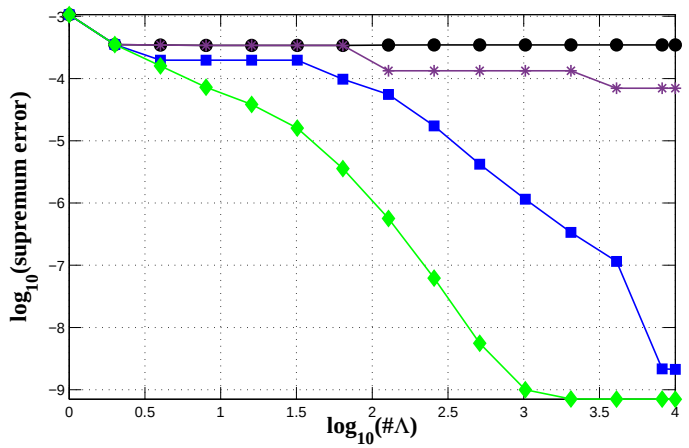


Adaptive vs non-adaptive

Adaptive methods are known to converge better than non-adaptive ones, but their analysis is more difficult.

A test case for linear-affine model in dimension $d = 64$: comparison between the approximation performance with Λ_n given by standard choices $\{\sup \nu_j \leq k\}$ (black) or $\{\sum \nu_j \leq k\}$ (purple) and by anisotropic choices based on a-priori bounds (blue) or adaptively generated (green).



Highest polynomial degree for Λ_{1000} with different choices : 1, 2, 162 and 114.

Downward closed index sets

For adaptive algorithms it is critical that the index chosen sets are downward closed

$$\nu \in \Lambda \text{ and } \mu \leq \nu \Rightarrow \mu \in \Lambda,$$

where $\mu \leq \nu$ means that $\mu_j \leq \nu_j$ for all $j \geq 1$.

Such sets are also called lower sets. This property does not generally holds for the sets corresponding to the n largest estimates, however the same convergence rates as proved in the approximation theorems, can be proved when imposing such a structure.

If Λ is downward closed, we consider the polynomial space

$$\mathbb{P}_\Lambda = \operatorname{span}\{y \to y^\nu \,:\, \nu \in \Lambda\} = \operatorname{span}\{L_\nu \,:\, \nu \in \Lambda\} = \operatorname{span}\{H_\nu \,:\, \nu \in \Lambda\}$$

and its V-valued version

$$V_\Lambda := \{\sum_{\nu \in \Lambda} v_\nu y^\nu \,:\, v_\nu \in V\} = V \otimes \mathbb{P}_\Lambda.$$

After having selected Λ_n we search for a computable approximation of u in V_{Λ_n}.

Note that $\dim(V_{\Lambda_n}) = \infty$. In practice we use $V_{\Lambda_n,h} = V_h \otimes \mathbb{P}_{\Lambda_n}$ which has dimension

$$\dim(V_{\Lambda_n,h}) = \dim(V_h)\dim(\mathbb{P}_{\Lambda_n}) = N_h n < \infty.$$

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$$\mathbb{P}_\Lambda = \operatorname{span}\{y \to y^\nu \,:\, \nu \in \Lambda\} = \operatorname{span}\{L_\nu \,:\, \nu \in \Lambda\} = \operatorname{span}\{H_\nu \,:\, \nu \in \Lambda\}$$

and its V-valued version

$$V_\Lambda := \{\sum_{\nu \in \Lambda} v_\nu y^\nu \,:\, v_\nu \in V\} = V \otimes \mathbb{P}_\Lambda.$$

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Note that $\dim(V_{\Lambda_n}) = \infty$. In practice we use $V_{\Lambda_n,h} = V_h \otimes \mathbb{P}_{\Lambda_n}$ which has dimension

$$\dim(V_{\Lambda_n,h}) = \dim(V_h)\dim(\mathbb{P}_{\Lambda_n}) = N_h n < \infty.$$

Downward closed index sets

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Strategies to build the polynomial approximation : intrusive methods

1. Galerkin method : based on a space-parameter variational form (test the parametric PDE on arbitrary $y \mapsto v(y)$ and integrate both in x and y). Example for model 0 : find $u \in L^2(U, V, \mu)$ such that for all $v \in L^2(U, V, \mu)$,

$$A(u, v) := \int_U \int_D a(x, y) \nabla u(x, y) \nabla v(x, y) dx d\mu(y) = \int_U \langle f, v(y) \rangle d\mu(y) =: L(v),$$

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$$\|u - u_n\|_{L^2(U,V,\mu} \leq (R/r)^{1/2} \min_{v \in V_{\Lambda_n}} \|u - v\|_{L^2(U,V,\mu}.$$

After space discretization, Galerkin problem in $V_{\Lambda_n,h}$ gives a $(nN_h) \times (nN_h)$ system.

2. Exact computation of the Taylor coefficients $\|t_\nu\|_V$, based on the recursive formula.

After space discretization, sequence of n systems of size $N_h \times N_h$.

Adaptive algorithms with optimal theoretical guarantees exist for both method 1 (Gittelson-Schwab) and 2 (Chkifa-Cohen-DeVore-Schwab).

These methods apply to other models, however mainly confined to linear PDEs, with affine parameter dependence.

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Exact adaptive computation of the Taylor coefficients

With e_j the Kroenecker sequence of index j,

$$\int_D \bar{a} \nabla t_\nu \nabla v = - \sum_{j:\ \nu_j \neq 0} \int_D \psi_j \nabla t_{\nu - e_j} \nabla v, \quad v \in V.$$

If Λ_n is downward closed, this allows us to compute all t_ν by recursively solving n boundary value problems, or $N_h \times N_h$ systems after space discretization in V_h.

Adaptive method : start with $\Lambda_1 = \{0\}$. Given that we have computed Λ_k and the $(t_\nu)_{\nu \in \Lambda_k}$ we compute the t_ν for ν in the margin

$$\mathcal{M}(\Lambda_k) = \mathcal{M}_k := \{\nu \notin \Lambda_k\,;\, \nu - e_j \in \Lambda_k \ \text{ for some } j\},$$

and build the new set by bulk search : choose $\Lambda_{k+1} = \Lambda_k \cup \mathcal{S}_k$, with $\mathcal{S}_k \subset \mathcal{M}_k$ smallest such that $\sum_{\nu \in \mathcal{S}_k} \|t_\nu\|_V^2 \geq \theta \sum_{\nu \in \mathcal{M}_k} \|t_\nu\|_V^2$, for a fixed $\theta \in]0,1[$.

Key property (saturation) : under (UEA), for any lower set Λ there exists a constant C such that

$$\sum_{\nu \notin \Lambda} \|t_\nu\|_V^2 \leq C \sum_{\nu \in \mathcal{M}(\Lambda)} \|t_\nu\|_V^2.$$

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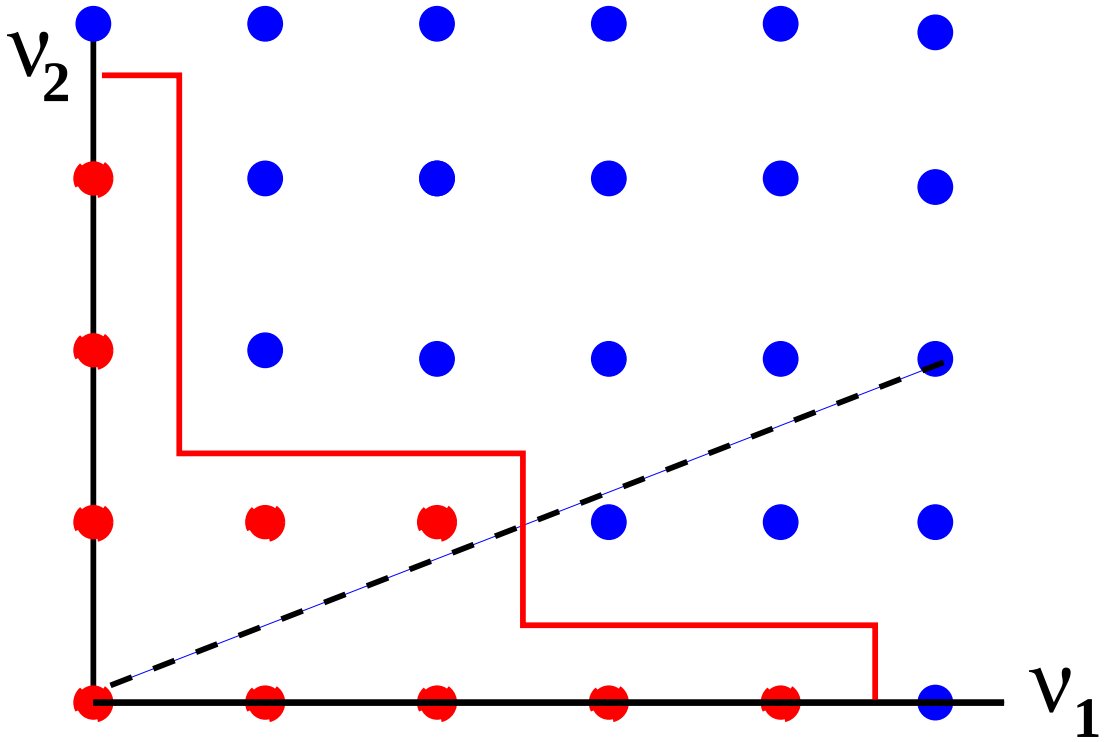
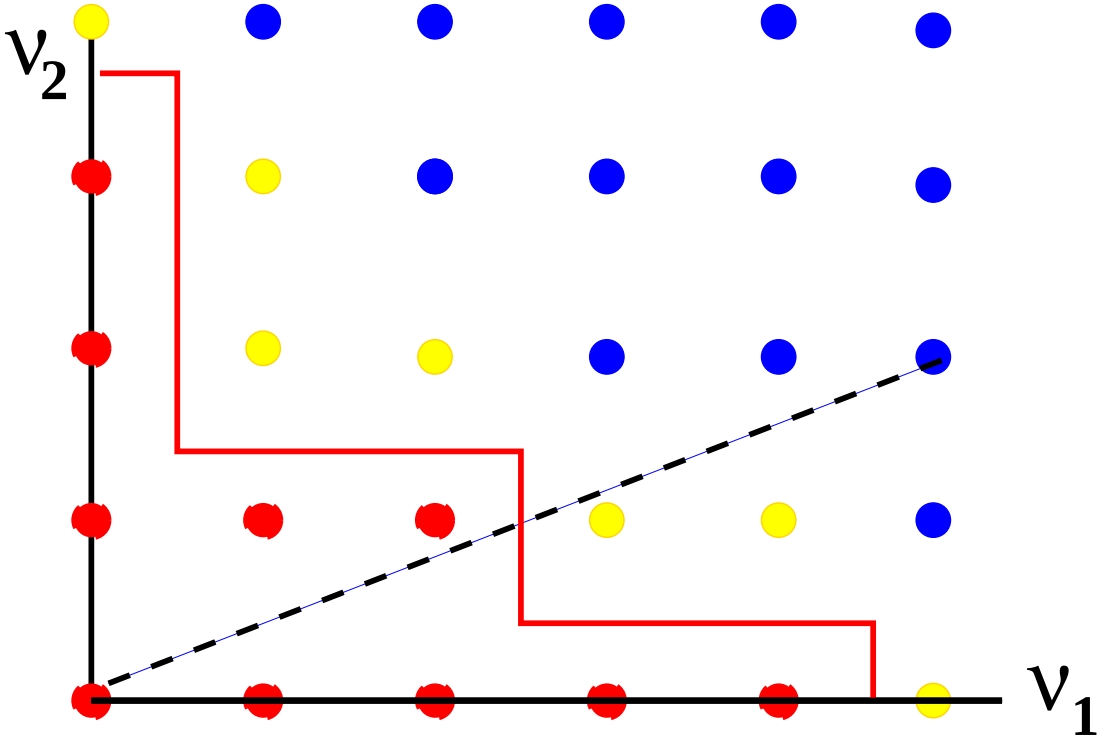

ν_2
ν_1


ν_2
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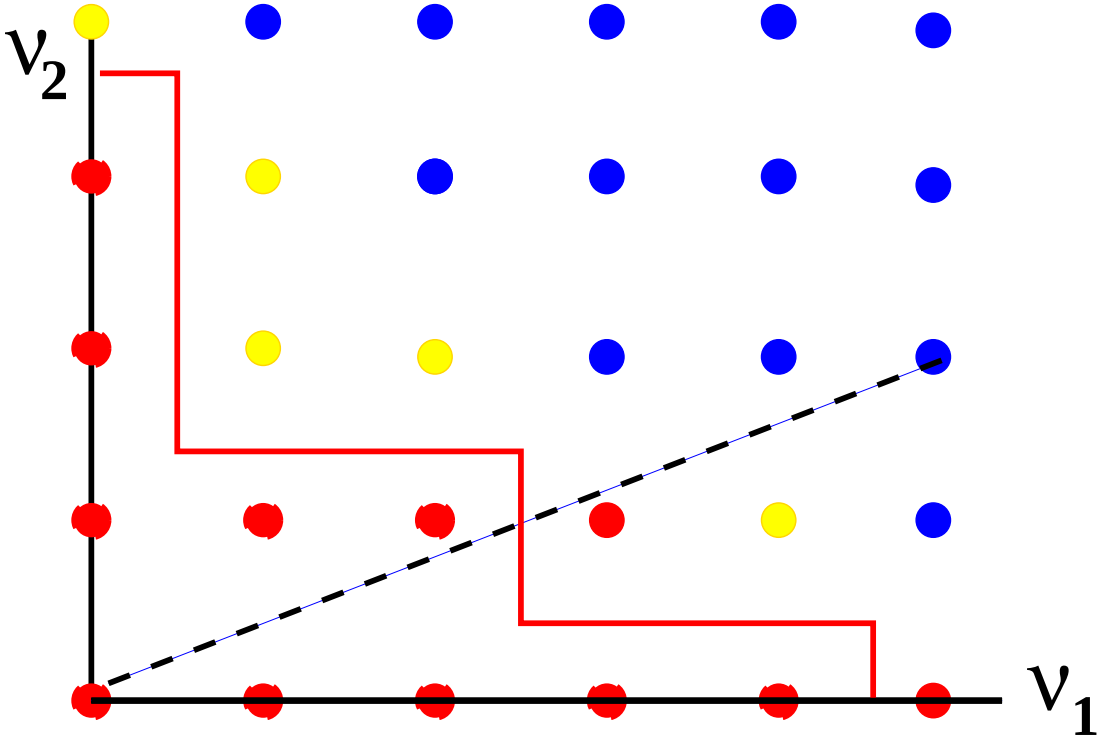
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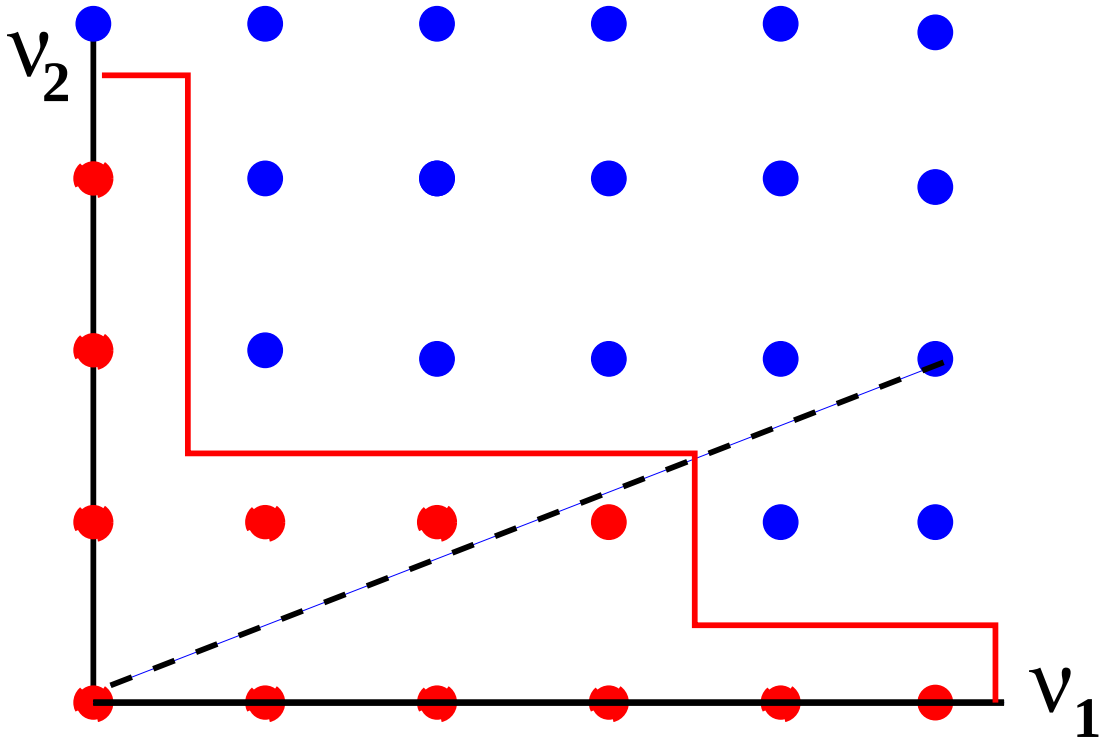
ν_2
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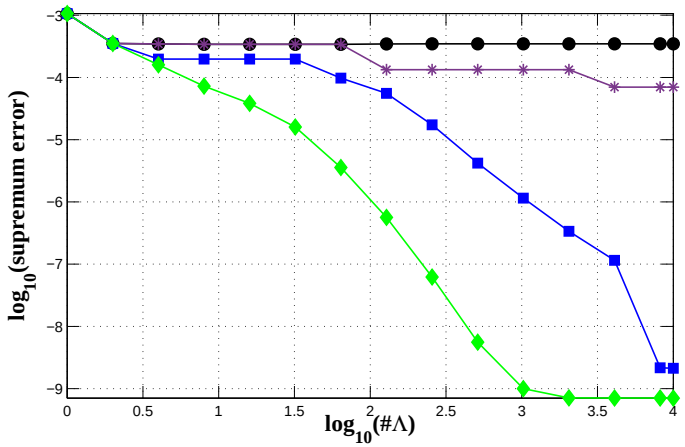
Test case in high dimension $d = 64$

Physical domain $D = [0,1]^2 = \cup_{j=1}^{d} D_j$.

Diffusion coefficients $a(x,y) = 1 + \sum_{j=1}^{d} y_j \left(\frac{0.9}{j^2}\right) \chi_{D_j}$. Thus $U = [-1,1]^{64}$.

Adaptive search of Λ implemented in C++, spatial discretization by FreeFem++.

Comparison between the Λ_k generated by the adaptive algorithm (green) and non-adaptive choices $\{\sup \nu_j \leq k\}$ (black) or $\{\sum \nu_j \leq k\}$ (purple) or k largest a-priori bounds on the $\|t_\nu\|_V$ (blue).



Highest polynomial degree with $\#(\Lambda) = 1000$ coefficients : 1, 2, 162 and 114.

Computation of the average solution

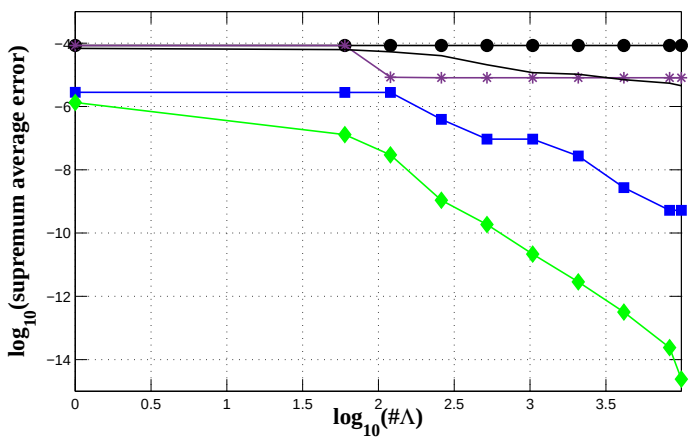
Assuming that y is uniformly distributed on $U = [-1,1]^{64}$, we compute the average solution

$$\bar{u} = \mathbb{E}(u),$$

either by the deterministic approach

$$\bar{u}_\Lambda := \sum_{\nu \in \Lambda} t_\nu \mathbb{E}(y^\nu), \quad \mathbb{E}(y^\nu) = \prod_{j>0} \Big(\int_{-1}^{1} t^{\nu_j} \frac{dt}{2} \Big) = \prod_{j>0} \frac{1 + (-1)^{\nu_j}}{2 + 2\nu_j},$$

or by the Monte Carlo approach $\bar{u}_n := \frac{1}{n} \sum_{i=1}^{n} u(y^i)$, where $y^1, \cdots, y^n$ are n independent realization of y.



Error curves in terms of number of solved bvp (MC in full line).

Strategies to build the polynomial approximation : non-intrusive methods

Based on snapshots $u_i := u(y^i)$ for $i = 1, \dots, m$..

1. Pseudo spectral methods : computation of $\sum_{\nu \in \Lambda_n} v_\nu L_\nu$ by quadrature

$$v_\nu = \int_U u(y) L_\nu(y) d\mu(y) \approx \sum_{i=1}^m w_i u(y^i) L_\nu(y^i).$$

2. Interpolation : with $m = n = \#(\Lambda_n) = \dim(\mathbb{P}_{\Lambda_n})$ search for a unique polynomial $u_n = I_{\Lambda_n} u \in V_{\Lambda_n}$ such that

$$u_n(y^i) = u_i, \quad i = 1, \dots, n.$$

3. Least-squares : with $m \geq n$, search for polynomial $u_n \in V_{\Lambda_n}$ minimizing

$$\sum_{i=1}^m \|u_i - u_n(y^i)\|_V^2.$$

4. Underdetermined least-squares : with $m < n$ search for a polynomial $u_n \in V_{\Lambda_n}$ minimizing

$$\sum_{i=1}^m \|u_i - u_n(y^i)\|_V^2 + \pi(u_n),$$

where π is a penalization functional. Compressed sensing : take for π the (weighted) ℓ^1 sum of V-norms of Legendre coefficients of u_n (promote sparse solutions).

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2. Interpolation : with $m = n = \#(\Lambda_n) = \dim(\mathbb{P}_{\Lambda_n})$ search for a unique polynomial $u_n = I_{\Lambda_n} u \in V_{\Lambda_n}$ such that

$$u_n(y^i) = u_i, \quad i = 1, \dots, n.$$

3. Least-squares : with $m \geq n$, search for polynomial $u_n \in V_{\Lambda_n}$ minimizing

$$\sum_{i=1}^m \|u_i - u_n(y^i)\|_V^2.$$

4. Underdetermined least-squares : with $m < n$ search for a polynomial $u_n \in V_{\Lambda_n}$ minimizing

$$\sum_{i=1}^m \|u_i - u_n(y^i)\|_V^2 + \pi(u_n),$$

where π is a penalization functional. Compressed sensing : take for π the (weighted) ℓ^1 sum of V-norms of Legendre coefficients of u_n (promote sparse solutions).

Advantages of intrusive methods

Applicable to a broad range of models, in particular non-linear PDEs.

Adaptive algorithms seem to work well for the interpolation and least squares approach, however with no theoretical guarantees.

Additional prescriptions for non-intrusive methods :

(i) Progressive : enrichment $\Lambda_n \longrightarrow \Lambda_{n+1}$ requires only one or a few new snapshots.

(ii) Stable : moderate growth with n of the Lebesgue constant relative to the interpolation operator.

Main issue : how to best choose the point y^i ?

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A commonly used non-polynomial method : RKHS interpolation

Given a set of point $\{y^1, \ldots, y^n\}$, there are infinitely many functions that admit the values $\{u(y^1), \ldots, u(y^n)\}$ at these points.

Some a-priori information needs to be injected in order to make a choice. One way to do this is through the minimization of a certain energy among all possible candidates.

Remark : in the univariate case, the piecewise linear and cubic spline interpolants on an interval I minimize the elastic and torsion energies $\int_I |v''|^2$ and $\int_I |v''''|^2$, respectively.

Reproducing Kernel Hilbert Space (RKHS) : a Hilbert space of function $\mathcal{H}$ defined on some domain U that is continuously embedded in the space of continuous function $C(U)$ (in our case $U = [-1, 1]^d$).

We assume that the space is rich enough such that for all $\{y^1, \ldots, y^n\}$ and values $\{v_1, \ldots, v_n\}$ there exists $v \in \mathcal{H}$ such that $v(y^i) = v_i$ for $i = 1, \ldots, n$.

Example : Sobolev space $H^s(U)$ with $s > d/2$.

RKHS interpolation (Kimmeldorf-Wahba, 1971, Duchon, 1977) : define interpolant as

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The reproducing kernel

For any $y \in U$, there exists $K_y \in \mathcal{H}$ such that

$$\langle K_y, v \rangle_X = v(y), \quad v \in \mathcal{H}.$$

The functions $(K_y)_{y \in U}$ are complete in $\mathcal{H}$. We define the reproducing kernel (RK) as

$$K(y,z) := \langle K_y, K_z \rangle_{\mathcal{H}} = K_y(z) = K_z(y)$$

The RK satisfies the positive definiteness property

$$\sum_{i=1}^{n} \sum_{j=1}^{n} K(y^i, y^j) c_i c_j > 0, \quad (c_1, \ldots, c_n) \neq (0, \ldots, 0), \quad y^1, \ldots, y^n \in U, \quad n \geq 0.$$

Conversly (Aronszajn, 1950), a function K satisfying this property generates a RKHS $\mathcal{H} = \mathcal{H}_K$ defined as the closure of the linear combinations of the functions $K_y = K(y, \cdot)$ for the norm induced by the inner product $\langle K_y, K_z \rangle_{\mathcal{H}} := K(y,z)$.

Radial basis functions (RBF) : if RK is of the form $K(y,z) = k(|y-z|)$, the functions K_y is the translate at y of the radial function $z \mapsto k(|z|)$ (e.g. Gaussian $e^{-a|z|^2}$).

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It is then easily seen that the RKHS interpolation is of the form $I_n u = \sum_{j=1}^n c_j K_{y^j}$, where $(c_1, \dots, c_n)$ is the unique solution to the system

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For a given positive definite kernel K we consider a centered gaussian process v with covariance $K(y, z)$ (reflects the uncertainty on the unknown u).

Then I_n can be defined as the conditional expectation

$$I_n(y) = \mathbb{E}\Big(v(y) \;\Big|\; v(y^j) = u(y^j),\ j = 1, \dots, n \Big).$$

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Adaptive strategies

The gaussian process interpretation leads to natural strategies for adaptive algorithms :

1. Point selection : given $y^1, \dots, y^n$, choose y^{n+1} where $\mathbb{E}(|u(y) - I_n u(y)|^2)$ is largest.

2. Kernel adaptation : using cross-validation, for example with anisotropic gaussians

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find b which minimizes $\sum_{i=1}^{n} \left| u(y^i) - I_{\{y^1,\dots,y^n\}-\{y^i\}} u(y^i) \right|^2$

Not much is known on the analysis of these stragegies (need relevant model classes).

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We assume that Λ is a lower set :

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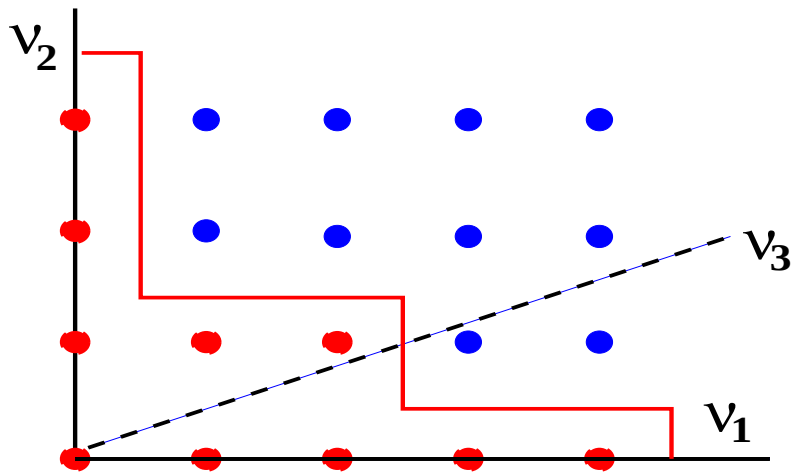

Motivation : for relevant classes of functions u arising from parametric PDEs, there exists sequences of lower sets $(\Lambda_n)_{n\geq 0}$ such that for some $s > 0$,

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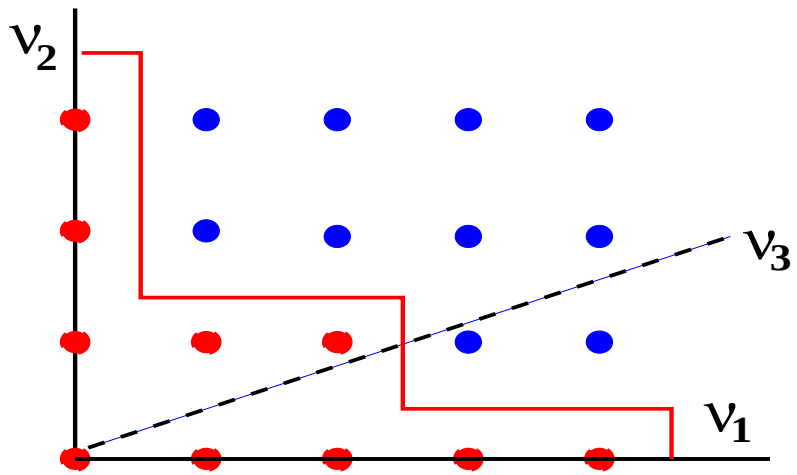

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Univariate nested interpolation

Let $\{t_0, t_1, t_2 \ldots\}$, be an infinite sequence of pairwise distinct points in $[-1, 1]$ and let I_k be the univariate interpolation operator on $\mathbb{P}_k$ associated to the section $\{t_0, \ldots, t_k\}$.

Hierarchical (Newton) form :

$$I_k = \sum_{l=0}^{k} \Delta_l, \quad \Delta_l := I_l - I_{l-1} \quad \text{and} \quad I_{-1} := 0.$$

Note that $\Delta_k \mathbb{P}_l = 0$ and $\Delta_k u(t_l) = 0$ for all $l < k$. Expansion in a hierarchical basis

$$\Delta_l u = \alpha_l h_l, \quad \alpha_l := u(t_l) - I_{l-1} u(t_l) \text{ and } h_l(t) = \prod_{j=0}^{l-1} \frac{t - t_j}{t_l - t_j}.$$

The choice of $\{t_0, t_1, t_2 \ldots\}$ is important for stability. The usual choices, such as Chebychev or Clemshaw-Curtis, are not section of a single infinite sequence.

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Tensorization

Tensorized grid : for any multi-index ν, we define the point

$$z_\nu := (t_{\nu_1}, t_{\nu_2}, \dots) \in U.$$

Tensorized operators : for any multi-index μ, we define

$$I_\mu = \otimes_{j\geq 1} I_{\mu_j} \quad \text{and} \quad \Delta_\mu := \otimes_{j\geq 1} \Delta_{\mu_j}.$$

I_μ is the interpolation operator on the space of polynomials of degree μ_j in each y_j

$$\mathbb{P}_\mu = \mathbb{P}_{R_\mu}, \quad R_\mu = \{\nu \ : \ \nu \leq \mu\},$$

associated to the grid of point

$$\Gamma_{R_\nu} := \{z_\nu \ : \ \nu \in R_\mu\}.$$

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Sparsification

Theorem (Cohen-Chkifa-Schwab, 2013, Dyn-Floater, 2013, Kuntzmann, 1959) : if Λ is any lower set, the grid

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is unisolvent for $\mathbb{P}_\Lambda = \mathrm{span}\{y \mapsto y^\nu \ : \ \nu \in \Lambda\}$ and the interpolant is given by

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Proof : Γ_Λ has the right cardinality, it suffices to prove that $I_\Lambda u(z_\mu) = u(z_\mu)$ for any $\mu \in \Lambda$. This follows from

$$I_\Lambda = I_\mu + \sum_{\nu \in \Lambda \ \nu \not\leq \mu} \Delta_\nu.$$

and observe that $I_\mu u(z_\mu) = u(z_\mu)$ and $\Delta_\nu u(z_\mu) = 0$ if $\nu \not\leq \mu$.

Sparsification

Theorem (Cohen-Chkifa-Schwab, 2013, Dyn-Floater, 2013, Kuntzmann, 1959) : if Λ is any lower set, the grid

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Hierarchical computation

With the tensorized hierarchical basis $H_\nu(y) = \prod_{j\geq 1} h_{\nu_j}(y_j)$, we have

$$\Delta_\nu u(y) = \alpha_\nu H_\nu(y).$$

where the coefficients α_ν can be computed recursively.

Write $\Lambda = \Lambda_n = \{\nu^1, \ldots, \nu^n\}$ where the enumeration is such that $\Lambda_k = \{\nu^1, \ldots, \nu^k\}$ is downward closed for all $k = 1, \ldots, n$. Then

$$\alpha_{\nu^k} = u(z_{\nu^k}) - I_{\Lambda_{k-1}} u(z_{\nu^k}).$$

Remark : the same general principles (tensorization, sparsification, hierarchical computation) apply to any other systems such as trigonometric polynomials or hierarchical piecewise linear finite elements.

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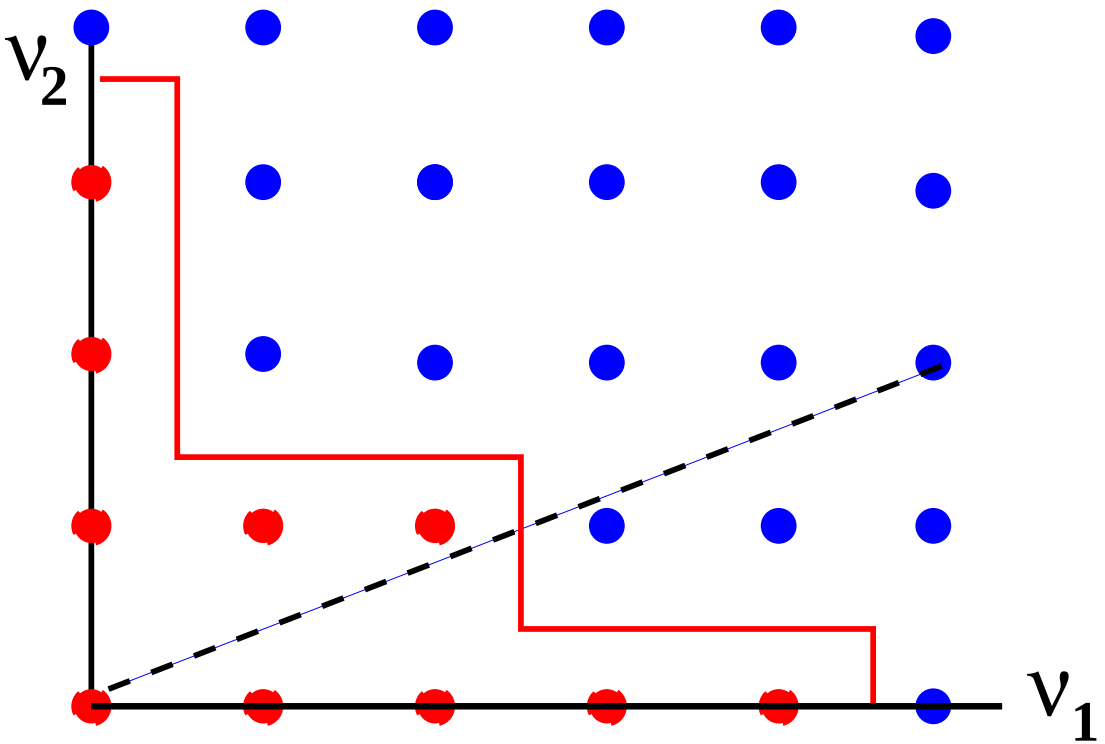
Adaptive algorithms

Given Λ, we consider its set of neighbors $\mathcal{N}(\Lambda)$ consisting of those $\nu \notin \Lambda$ such that $\Lambda \cup \{\nu\}$ is also a lower set.

Adaptive algorithm : given Λ_n, define $\Lambda_{n+1} := \Lambda_n \cup \{\nu^*\}$ with

$$\nu^* := \operatorname{argmax}\{\|\Delta_\nu u\|_{L^\infty} \ : \ \nu \in \mathcal{N}(\Lambda_n)\}.$$



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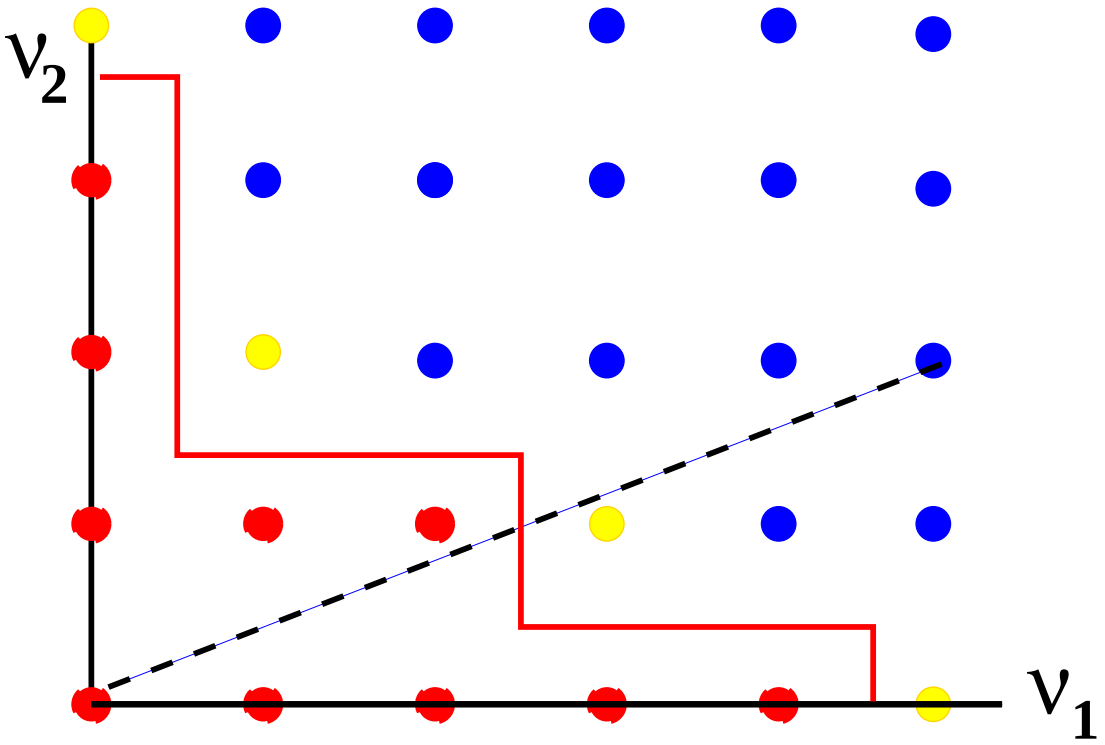

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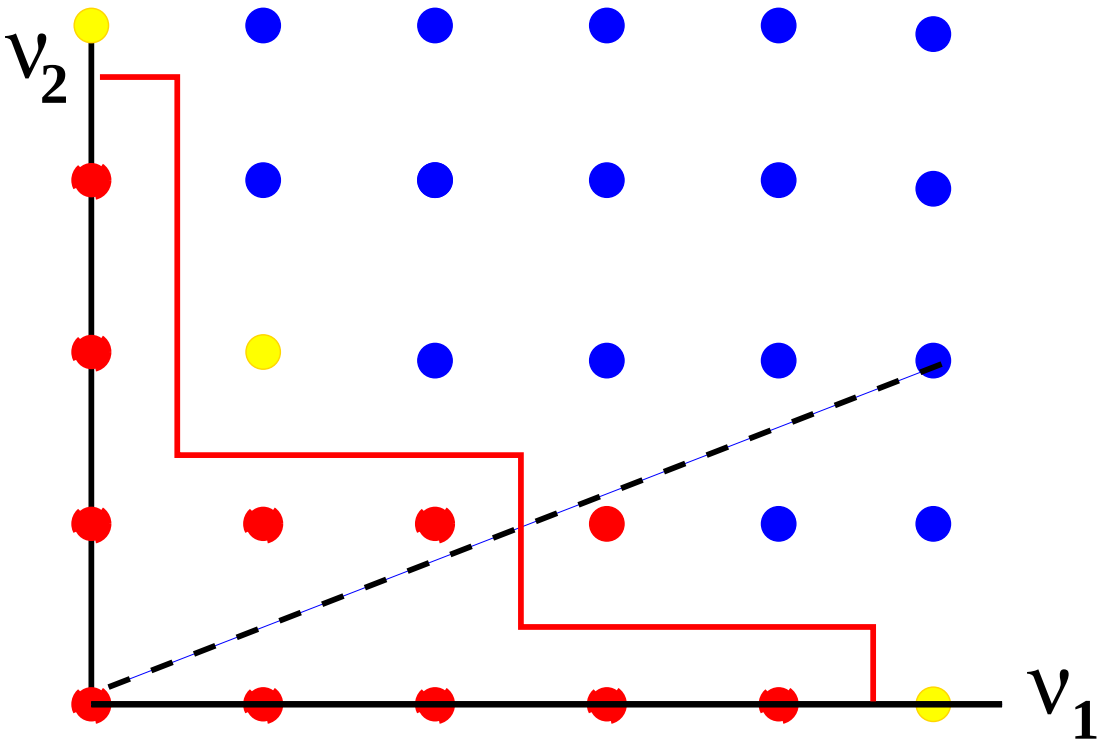

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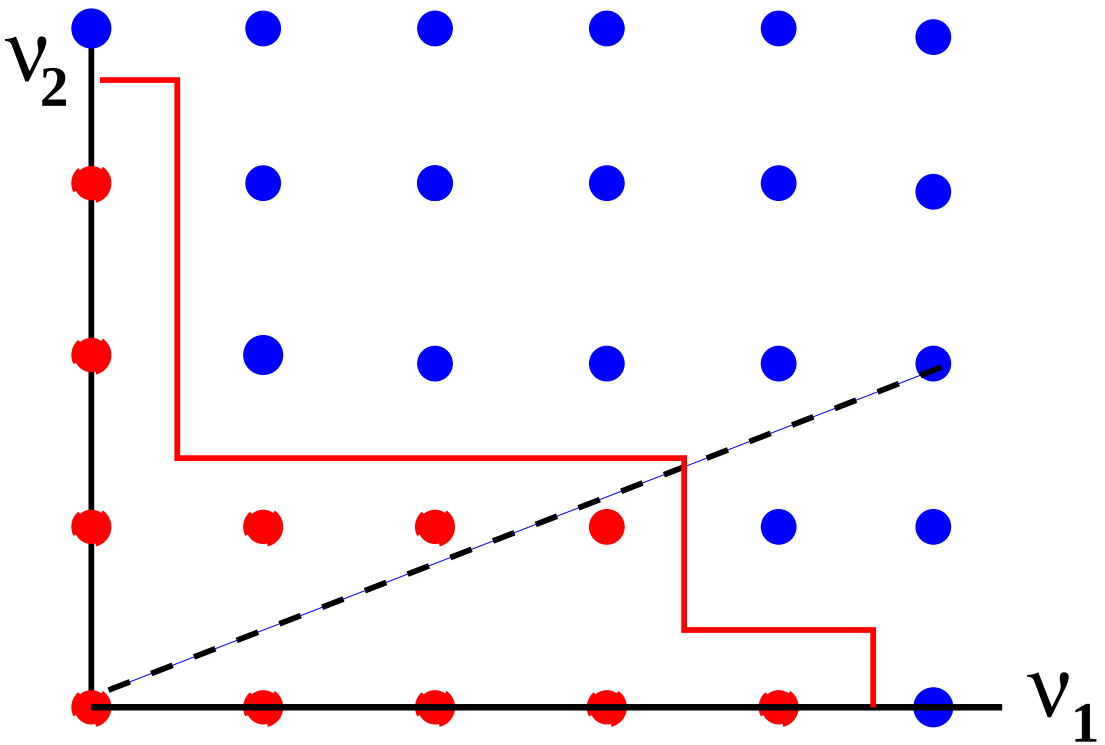

Theoretical difficulties

The previous adaptive algorithm may fail to converge (in particular $\Delta_\nu u = 0$ for some ν and $\Delta_\mu u \neq 0$ for a $\mu \geq \nu$.

Behaves well in many practical situations.

More conservative variant : Use the above selection rule if n is even, and for odd n choose $\nu^* \in \mathcal{N}(\Lambda_n)$ which was already contained in $N(\Lambda_k)$ for the smallest value of k.

Other variants : measure $\Delta_\nu u$ in L^p norm, use $|\int_U \Delta_\nu u|$ (integration), or $\nu^* \in \mathcal{N}(\Lambda_n)$ minimizing $u(z_\nu)$ (optimization)...

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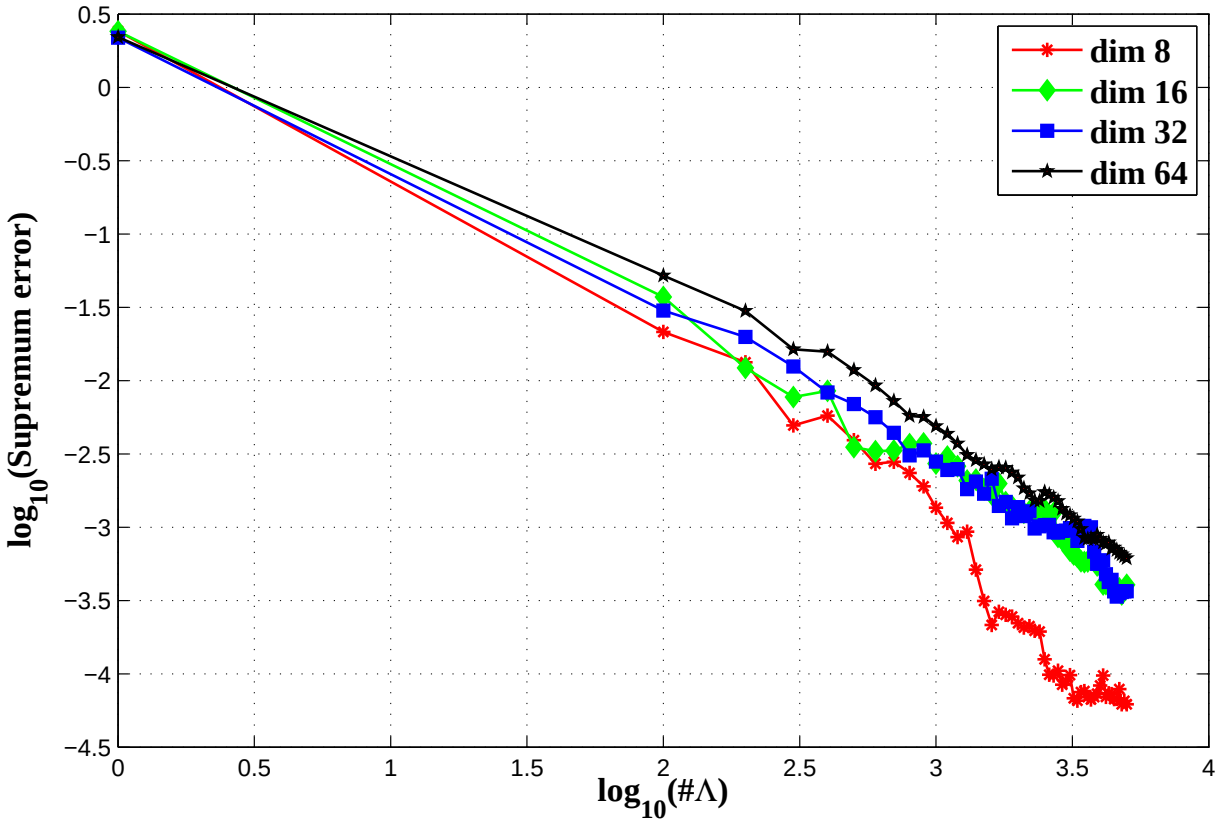
Robustness to dimension growth

We apply the adaptive interpolation algorithm to

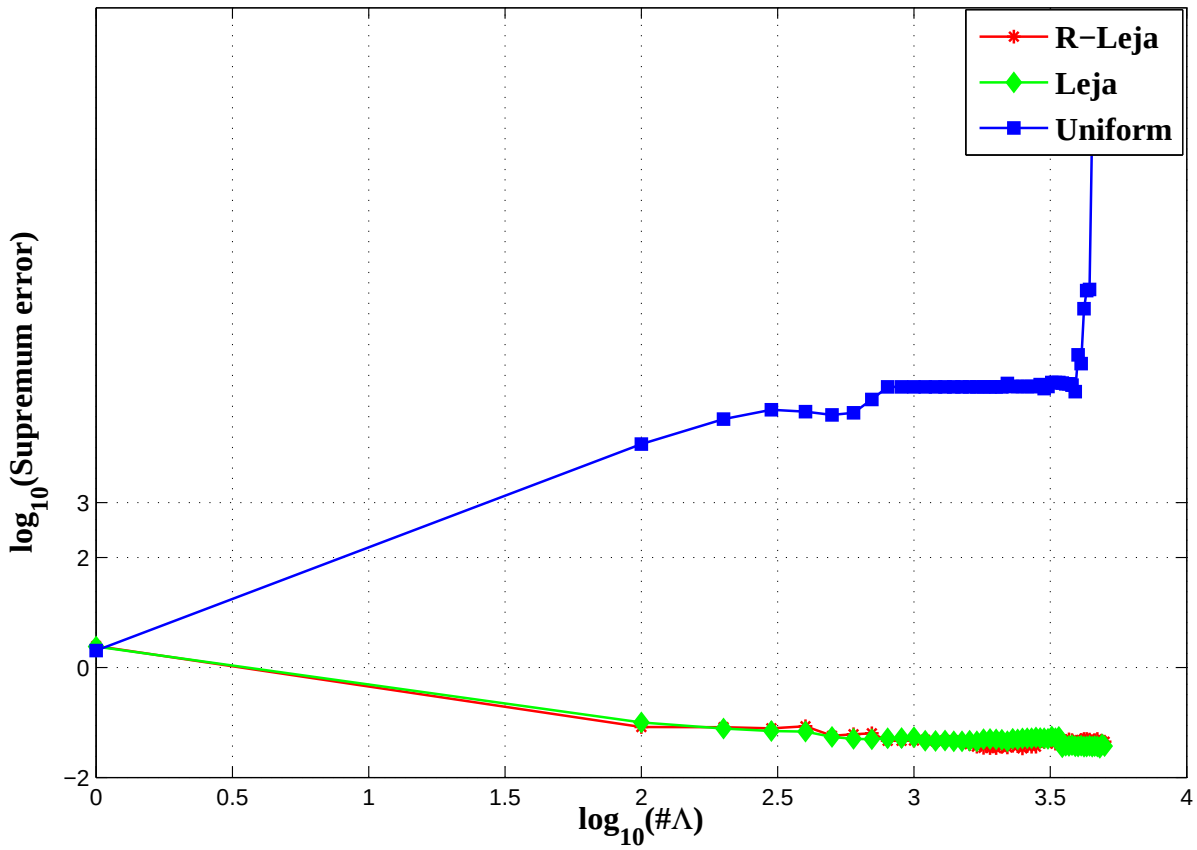
$$u(y) := \left(1 + \sum_{j=1}^{d} \gamma_j y_j\right)^{-1}, \quad \gamma_j = \frac{3}{5j^3},$$

for different numbers d of variables.



Robustness to noise

Same function u in dimension $d = 16$, with noisy samples (noise level $= 10^{-2}$). using adaptive interpolation based on different univariate sequences.



Stability

We want to study the Lebesgue constant

$$\mathbb{L}_\Lambda := \|I_\Lambda\|_{L^\infty \to L^\infty} = \sup_u \frac{\|I_\Lambda u\|_{L^\infty}}{\|u\|_{L^\infty}}$$

Useful for approximation since

$$\|u - I_\Lambda u\|_{L^\infty} \leq \|u - v\|_{L^\infty} + \|I_\Lambda v - I_\Lambda u\|_{L^\infty}, \quad v \in \mathbb{P}_\Lambda,$$

and thus

$$\|u - I_\Lambda u\|_{L^\infty} \leq (1 + \mathbb{L}_\Lambda) \min_{v \in \mathbb{P}_\Lambda} \|u - v\|_{L^\infty}$$

The following result relates $\mathbb{L}_\Lambda$ to the univariate Lebesgue constant

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Stability of univariate sequences

For Leja point, it is known (Taylor-Totik, 2008) that $\mathbb{L}_k$ is sub-exponential

$$\lim_{k \to +\infty} \mathbb{L}_k^{1/k} = 1.$$

Numerical computation seems to indicate that

$$\mathbb{L}_k \leq 1 + k.$$

Clemshaw-Curtis points $C_k = \{\cos(l\pi/k) \ : \ l = 0, \ldots, k\}$ are dyadically nested :

$$C_{2^j+1} \subset C_{2^{j+1}+1}.$$

For the values $k = 2^{j+1}$ we know that $\mathbb{L}_k \sim \log(k)$.

Problem : how to fill in the intermediate values ?

Sequencial enumeration : disastrous behaviour of $\mathbb{L}_k$.

Van der Corput enumeration : it can be proved (Chkifa, 2013) that

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This is also the projection of the Leja point for the complex unit disc (R-Leja points).

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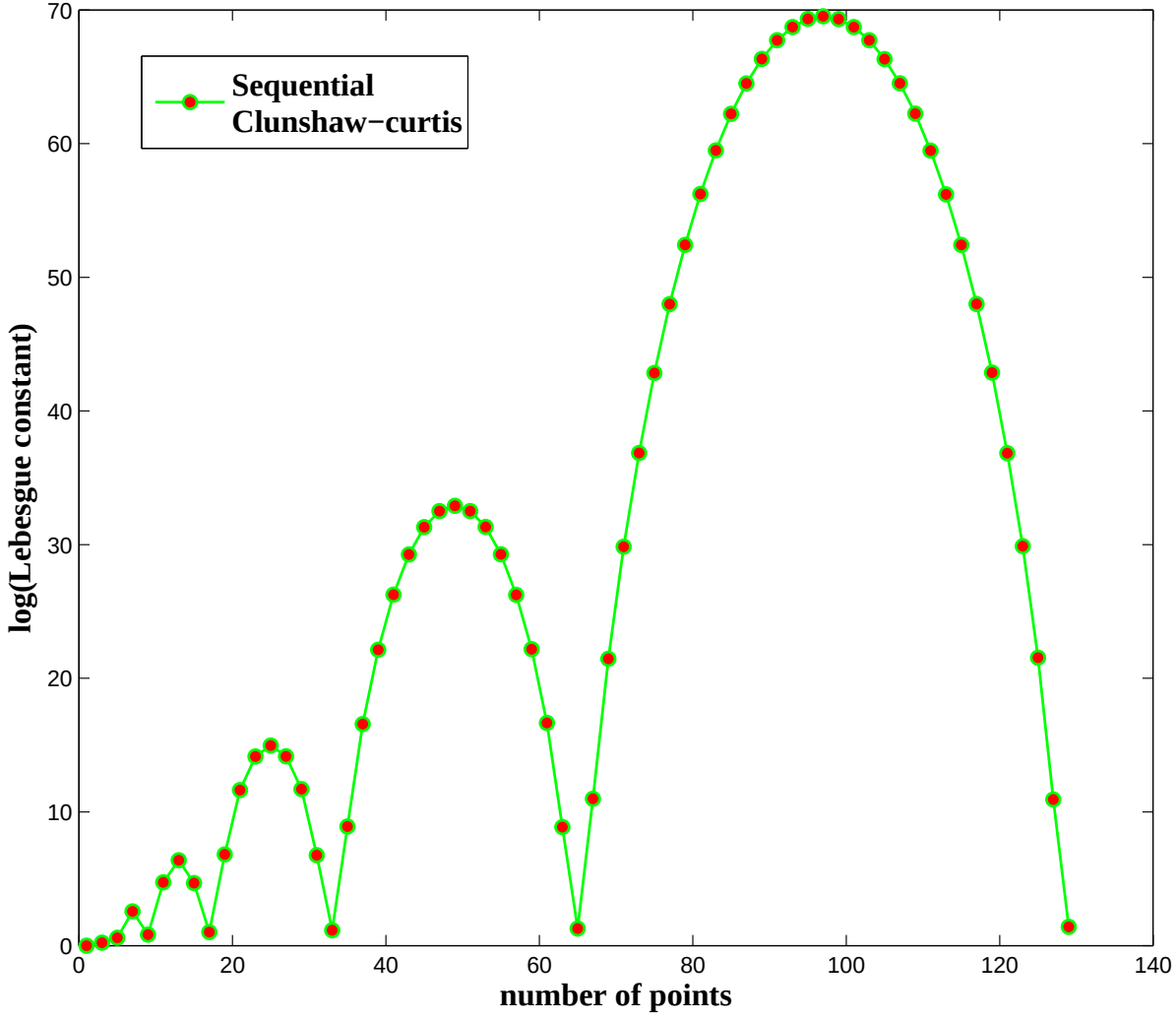
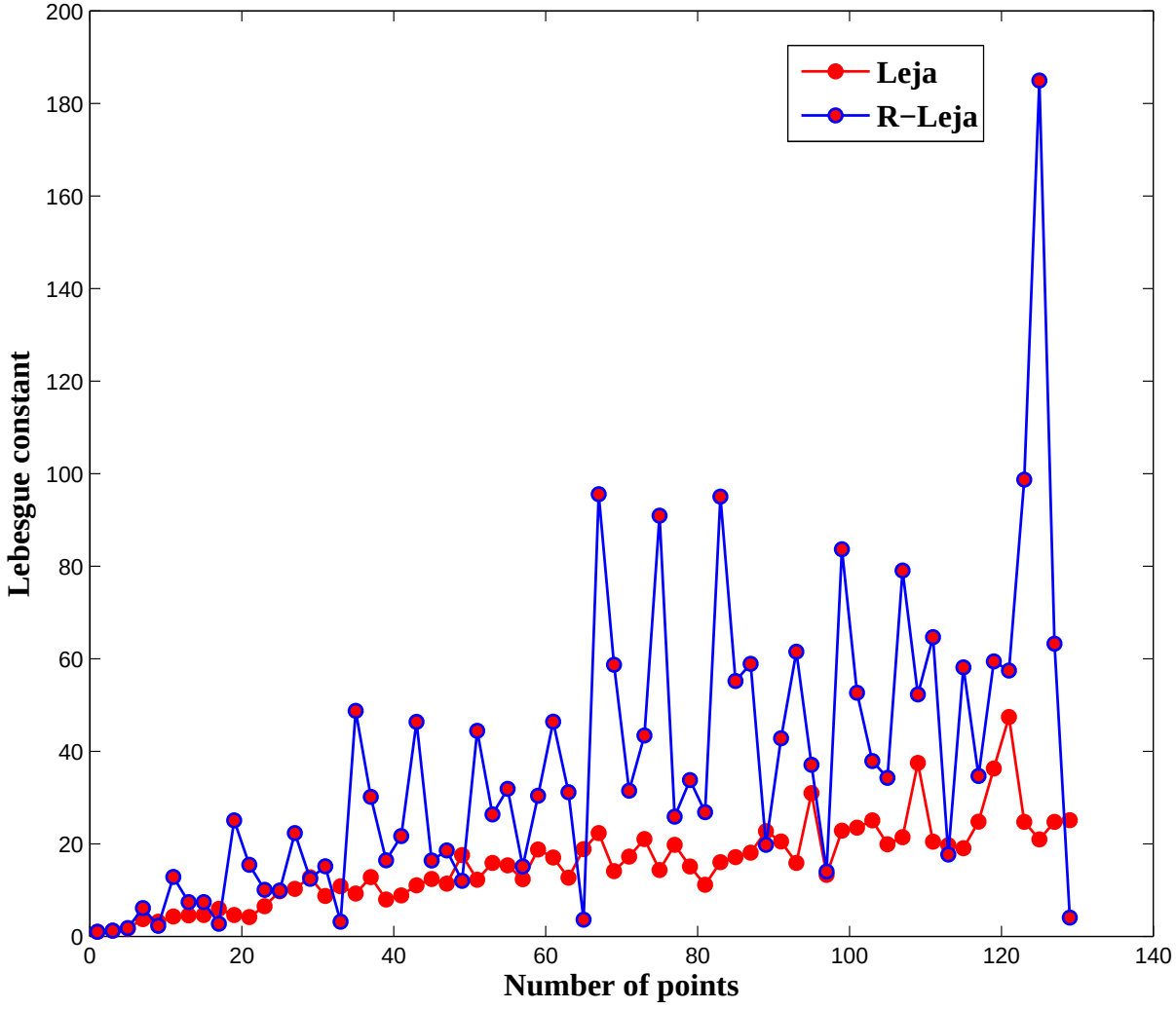
Lebesgue constant for the Clemshaw-Curtis point with sequencial intermediate filling.



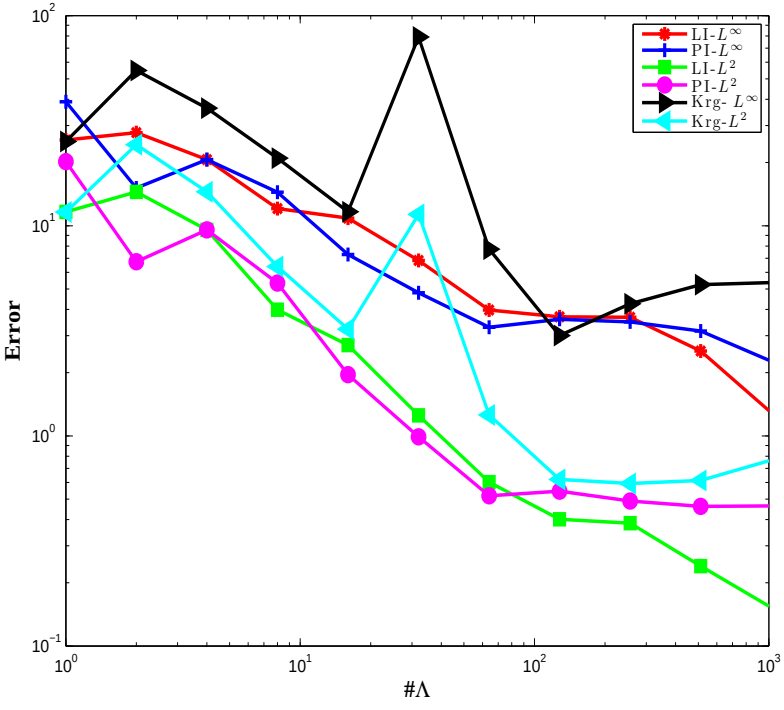
Stability

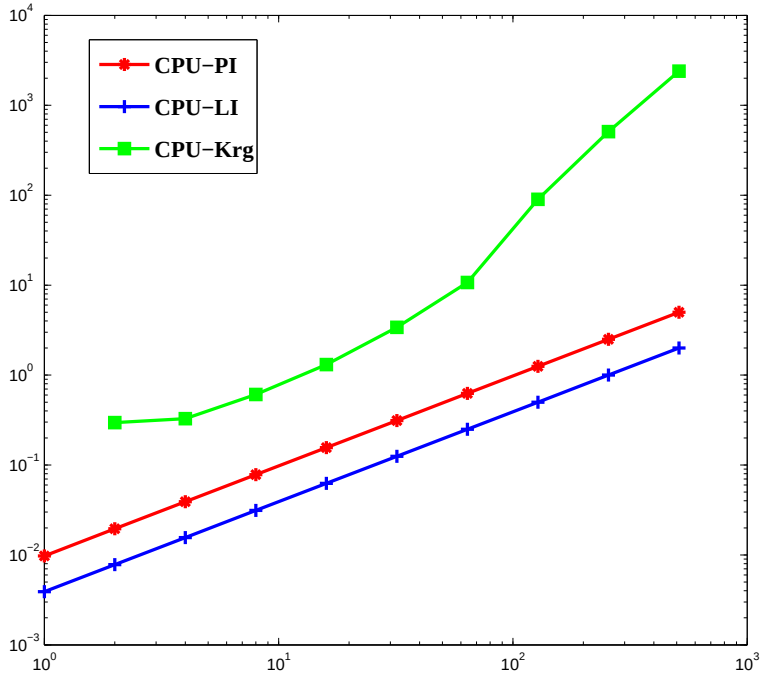
The Lebesgue constant for the Leja points (red) and the R-Leja points (blue).



Comparison with kriging interpolation algorithms

Test case : $y = (y_1, y_2, y_3, y_4, y_5)$ shape parameters in the design of an airfoil and $u(y)$ is the lift to drag ratio (scalar quantity of interest) obtained by ONERA numerical solver.





Error curves in terms of number of points are comparable.

The CPU cost for sparse interpolation scales linearly with the number of points.

This contrasts with kriging methods which require solving ill-conditionned linear systems of growing size + optimization of the parameters of a Gaussian kernel.

Reduced order modeling and n-width

Recall the benchmark of Kolmogorov n-width of the solution manifold

$$d_n(\mathcal{M})_V = \inf_{\dim(E)=n} \max_{v\in\mathcal{M}} \min_{w\in E} \|v-w\|_V = \inf_{\dim(E)=n} \max_{y\in U} \min_{w\in E} \|u(y)-w\|_V.$$

Uniform approximation estimates of the solution map $y \mapsto u(y)$ by polynomial (or other separable) expansions give an upper bound on n-width

$$d_n(\mathcal{M})_V \leq \min_{v\in V_{\Lambda_n}} \|u-v\|_{L^\infty(U,V)} \leq Cn^{-s}.$$

We do not know other approaches to estimate the n-width of the solution manifold by above.

These estimates might very pessimistic in the sense that he actual n-width $d_n(\mathcal{M})_V$ is much smaller than the right side.

We do not have results proving lower bounds for the n-widths of solution manifolds.

It is desirable to have numerical reduced modeling methods that can provably perform as good as the n-width benchmark.

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Reduced bases (Maday, Patera)

Define a reduced modeling space $V_n = \mathrm{span}\{u_1, \ldots, u_n\}$, where the u_i are particular instances (snapshots) from the solution manifold

$$u_i = u(a_i)$$

for some $a_1, \ldots, a_n \in \mathcal{M}$.

Greedy selection : having selected $u_1, \ldots, u_{k-1} \in \mathcal{M}$, choose the next instance by

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This algorithm is not realistic : $\|u(u) - P_{V_{k-1}} u(y)\|_V$ is unknown, however can be estimate at moderate cost by a-posteriori error analysis. Therefore, one rather apply a weak-greedy algorithm : u_k such that

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Comparison with n-width

Performance of reduced bases : $\sigma_n(\mathcal{M})_V := \max\{\|v - P_{V_n} v\|_V \ : \ v \in \mathcal{M}\}$

Comparison with n-width : $\sigma_n(\mathcal{M})_V$ can be much larger than $d_n(\mathcal{M})_V$ for an individual n and $\mathcal{M}$.

There exists $\mathcal{M}$ and n such that $\sigma_n(\mathcal{M})_V \geq 2^n d_n(\mathcal{M})_V$.

However, a more favorable comparison is possible in terms of convergence rates :

Theorem (Binev-Cohen-Dahmen-DeVore-Petrova-Wojtaszczyk, 2013) : For any $s > 0$ one has

$$\sup_{n \geq 1} n^s d_n(\mathcal{M})_V < \infty \Rightarrow \sup_{n \geq 1} n^s \sigma_n(\mathcal{M})_V < \infty,$$

and for any $a > 0$ there exists $b > 0$ such that

$$\sup_{n \geq 1} e^{an^s} d_n(\mathcal{M})_V < \infty \Rightarrow \sup_{n \geq 1} e^{bn^s} \sigma_n(\mathcal{M})_V < \infty.$$

Conclusions

The curse of dimensionality can be "defeated" by exploiting both smoothness and anisotropy in the different variables.

For certain models, this can be achieved by sparse polynomial approximations.

Adaptive algorithms with optimal theoretical guarantees are still to be developed, in particular for non-intrusive approaches (interpolation, collocation, least-squares).

Reduced bases achieve "almost" the same performance as optimal spaces corresponding to Kolmogorov n-width.