When is a system of Sylvester-type matrix equations well posed?

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Abstract

We study systems of Sylvester-type matrix equations, equations, i.e., systems of the form

$$\begin{aligned} A_1 X_{i_1}^{s_1} B_1 + C_1 X_{j_1}^{t_1} D_1 &= E_1, \\ A_2 X_{i_2}^{s_2} B_2 + C_2 X_{j_2}^{t_2} D_2 &= E_2, \\ &\dots \\ A_r X_{i_r}^{s_r} B_r + C_r X_{j_r}^{t_r} D_r &= E_r, \end{aligned}$$

where (for all $k = 1, 2, \dots, r$) the unknowns X_k and the coefficients A_k, B_k, C_k, D_k, E_k are $n \times n$ matrices, the indices i_k and j_k are in $\{1, 2, ..., r\}$, and each of the symbols s_k, t_k is either $1, T$ (for transpose) or H (for conjugate-transpose).

Simpler versions of this problem appear in several applications in systems and control theory, as well as in some algorithms in numerical linear algebra.

We are interested in determining for which values of A_k, B_k, C_k, D_k $k = 1, 2, \dots, r$ the system has a unique solution for each choice of the right-hand sides (unique solvability for each right-hand side, or well-posedness). In principle, this is equivalent to determining when a $rn^2 \times rn^2$ (or $2rn^2 \times 2rn^2$) matrix is invertible; however, we look for more efficient methods and more meaningful conditions. When the system is well-posed, we also look for a numerical method to solve it.

We make several successive transformations to reduce the problem to a simpler form, and then give a criterion to check well-posedness by working directly on the $n \times n$ coefficients. This procedure generalizes analogous criteria that exist for single equations and simpler cases, which are based on eigenvalues of suitable matrices and pencils. Our strategy is constructive, and can be turned into a $\mathcal{O}(rn^3)$ solution algorithm to solve well-posed systems.