Entropy solutions of degenerate parabolic equations

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In this talk we study entropy solutions to the degenerate parabolic equation

$$\partial_t u + \mathrm{div}_x\, f(u) = \mathrm{div}_x\big(a(u)\nabla u\big)\,, \quad (t,x) \in (0,+\infty) \times \mathbb{R}^d\,,$$

subject to the initial condition $u(0,\cdot) = u_0$. Here the degeneracy appears as the matrix $a(\lambda)$ is only positive semi-definite, i.e. it can be equal to zero in some directions and the directions can depend on λ. Equations of this form often occur in modelling flows in porous media and sedimentation-consolidation processes.

Since in the special case $a = 0$ (which is allowed) the equations becomes a (multidimensional) scalar conservation law, in the talk we start by recalling some known results in this setting (e.g. non-uniqueness of weak solutions (see Figure 1), notion of entropy solutions, existence of solutions and traces), and emphasising some open problems (e.g. heterogeneous case).

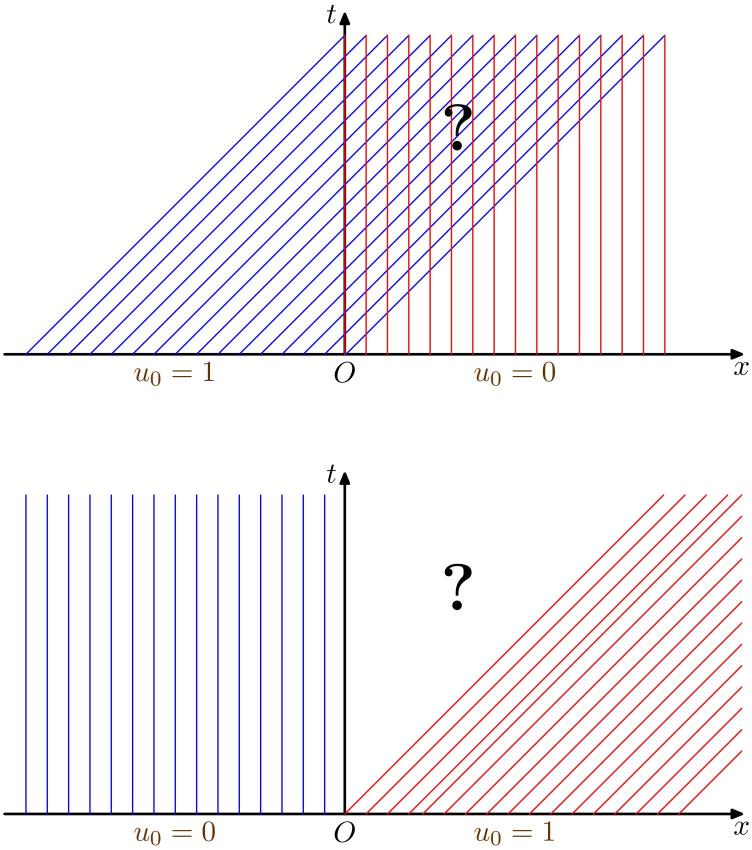

Figure 1: Visualisation of the characteristics for Burgers' equation ($d = 1$, $f(\lambda) = \lambda^2/2$, $a = 0$) for two initial data. On the first one we have a problem of intersecting characteristics, while on the second one the problem is how to uniquely extend the solution to the region not covered by any characteristic.

The notion of entropy solutions easily generalises for the degenerate parabolic equations, and it is well-established in the literature. Moreover, an equivalent kinetic formulation is also available, which will be used in this talk. Our contributions consist of obtaining the existence of entropy solutions for the starting problem by applying a suitable velocity averaging result, and proving that all such solutions admit the strong trace at $t = 0$. The latter results implies that the weak trace suffices for uniqueness and could be an important step into formulating the initial boundary value problem in the sense of Bardos, LeRoux and Nédélec.

This is joint work with Marin Mišur and Darko Mitrović.