Jensen-Steffensen Inequality: Accentuate the Negative

Milica Klaričić Bakula
Faculty of Science, University of Split

Let $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, be a convex function on I, and $\boldsymbol{x} = (x_1, \cdots, x_n) \in I^n$. If $\boldsymbol{p} = (p_1, \cdots, p_n)$ is a nonnegative real n-tuple such that $P_n = \sum_{i=1}^{n} p_i > 0$ then the well-known Jensen inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \tag{1}$$

holds.

To Steffensen's credit, it is known that the assumption "$\boldsymbol{p}$ is a nonnegative real n-tuple" can be relaxed at the expense of further restrictions on the n-tuple $\boldsymbol{x}$. Namely, if $\boldsymbol{x}$ is a monotonic (increasing or decreasing) n-tuple from I^n then for any real n-tuple $\boldsymbol{p}$ such that

$$0 \leq P_j = \sum_{i=1}^{j} p_i \leq P_n, \; j = 1, \cdots, n, \qquad P_n > 0, \tag{2}$$

we get

$$\overline{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \in I,$$

and (1) still holds. Inequality (1) under conditions (2) is known as the Jensen-Steffensen inequality for convex functions.

One can say that the Jensen-Steffensen inequality is "the ugly sister" of the Jensen inequality: not much admired and usually "not invited to the party". Our goal here is to show that "she" has many hidden beauties and that "she" can proudly walk hand in hand with her well-known sister.