An Algorithmic Approach to Cyclotomic Units

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For $n \in \mathbf{N}$ and $(a,n) = 1$ let $\epsilon_n := e^{\frac{2\pi i a}{n}}$ be a primitive nth unit root. We define the group of cyclotomic numbers $D^{(n)}$ as the multiplicative group generated by elements of the form $1 - \epsilon_n^k$ with $k \not\equiv 0 \bmod n$. The group of *cyclotomic units* is defined as $C^{(n)} := \mathbf{Z}[\epsilon_n]^* \cap D^{(n)}$. It is well known that the group of cyclotomic units is of finite index within the full unit group $\mathbf{Z}[\epsilon_n]^*$ of the cyclotomic extension $\mathbf{Q}(\epsilon_n)$. We extend the definitions above allowing $n = \infty$ with $D^{(\infty)} := \bigcup_{n\in\mathbf{N}} D^{(n)}$ and $C^{(\infty)} := \bigcup_{n\in\mathbf{N}} C^{(n)}$ respectively.

We investigate various aspects of multiplicative relations within $D^{(n)}$ and $C^{(n)}$ for $n \in \mathbf{N} \cup \{\infty\}$. This leads to a geometrical interpretation and visualization of these relations. Of particular interest are the so-called "Ennola" relations that occur when n is of the form $4pq$ or pqr where p, q and r are odd primes (with $n = 60$ and $n = 105$ being the smallest examples). We give explicit algorithms that construct these Ennola relations, determine a basis of both $D^{(n)}$ and $C^{(n)}$ and (recursively) calculate the basis representation of an arbitrary cyclotomic unit or number.