Multiparameter eigenvalue problems

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We will consider the *two-parameter eigenvalue problem*

$$\begin{aligned} A_1x_1 &= \lambda B_1x_1 + \mu C_1x_1, \\ A_2x_2 &= \lambda B_2x_2 + \mu C_2x_2, \end{aligned} \tag{1}$$

where A_i, B_i, and C_i are given $n_i \times n_i$ matrices over $\mathbb{C}$, $\lambda, \mu \in \mathbb{C}$, and $x_i \in \mathbb{C}^{n_i}$ for $i = 1, 2$. A pair (λ, μ) is an *eigenvalue* if it satisfies (1) for nonzero vectors x_1, x_2. The tensor product $x_1 \otimes x_2$ is then the corresponding *right eigenvector*. Similarly, $y_1 \otimes y_2$ is the corresponding *left eigenvector* if $0 \neq y_i \in \mathbb{C}^{n_i}$ and $y_i^*(A_i - \lambda B_i - \mu C_i) = 0$ for $i = 1, 2$.

On the tensor product space $S := \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ of the dimension $N := n_1n_2$ we can define

$$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2, \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2, \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2. \end{aligned}$$

The two-parameter problem (1) is *nonsingular* if its operator determinant Δ_0 is invertible. In this case $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ commute and problem (1) is equivalent to the associated problem

$$\begin{aligned} \Delta_1 z &= \lambda \Delta_0 z, \\ \Delta_2 z &= \mu \Delta_0 z \end{aligned} \tag{2}$$

for decomposable tensors $z \in S$, $z = x_1 \otimes x_2$.

In the first part a basic theory of the multiparameter eigenvalue problems will be presented together with some examples where such problems appear. In the second part some numerical methods for the two-parameter eigenvalue problem will be discussed. One possible approach is to solve the associated couple of generalized eigenproblems (2), but this is only feasible for problems of low dimension because the size of the matrices of (2) is $N \times N$. For larger problems we suggest a Jacobi–Davidson type method: a one-sided variant for the right definite case and a two-sided variant for a general nonsingular two-parameter eigenvalue problem.

References

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